Symmetric Products and a realization of generators in the cohomology of Polyhedral Products

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A report of joint work with

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and our late colleagues

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and the work of others.

A category

Begin by setting $[m] = \{1, 2, \ldots, m\}$ and define a category $\boldsymbol{\mathcal{C}([m])}$ as follows

objects: Pairs $(\underline{X}, \underline{A})$ where

$$(\underline{X}, \underline{A}) \;=\; \big\{(X_1, A_1), (X_2, A_2), \ldots, (X_m, A_m)\big\}$$

is a family of based CW-pairs.

morphisms:

$$\underline{f}\colon (\underline{X}, \underline{A}) \longrightarrow (\underline{Y}, \underline{B})$$

consisting of m continuous maps $f_i\colon X_i \longrightarrow Y_i$

satisfying $f_i(A_i) \subset B_i$.

A functor

Next, let K be a simplicial complex on the vertex set $[m]$.

A **polyhedral product** is a functor

$$Z(K;-)\colon \boldsymbol{\mathcal{C}([m])} \longrightarrow \mathbf{Top}$$

satisfying

$$Z(K;(\underline{X},\underline{A})) \;\subseteq\; \prod_{i=1}^{m} X_i$$

and is defined as a colimit of a certain diagram

$$D : K \to CW_*$$

A definition

For each $\sigma \in K$, the diagram

$$D : K \to CW_*$$

is defined by

$$D(\sigma) = \prod_{i=1}^{m} W_i, \quad \text{where} \quad W_i = \begin{cases} X_i & \text{if} \quad i \in \sigma \\ A_i & \text{if} \quad i \in [m] - \sigma. \end{cases}$$

We set

$$Z(K; (\underline{X}, \underline{A})) \; = \; \textbf{colim}\big(D(\sigma)\big).$$

Here, the colimit is a union of natural subspaces

$$\bigcup_{\sigma \in K} D(\sigma) \; \subseteq \; \prod_{i=1}^{m} X_i$$

(but the colimit structure is essential.)

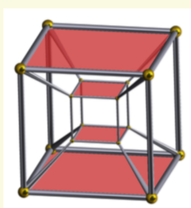
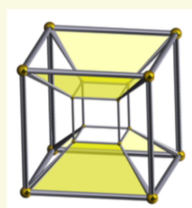
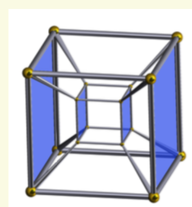
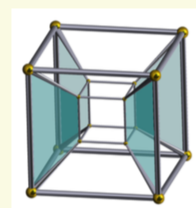
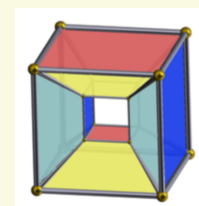
The 2-torus as a polyhedral product

We take K to be the boundary of a square, so $m = 4$.

$$(\underline{X}, \underline{A}) = \{(D^1, S^0)\}.$$

$$\Longrightarrow \quad Z(K; (D^1, S^0)) \subseteq D^1 \times D^1 \times D^1 \times D^1$$

is a subset of the 4-cube

$D(\sigma_1)$ $D(\sigma_2)$ $D(\sigma_3)$ $D(\sigma_4)$

This figure is from the thesis of
Alvise Trevisan

The Hopf map likes polyhedral products

To see the Hopf map in this setting, we take

$$K = \{\varnothing, v_1, v_2\} \text{ and } (X, A) = (D^2, S^1)$$

$$\begin{tikzcd}
S^3 \arrow[rr, "\eta"] \arrow[d, "\cong"] & & S^2 \\
\partial(D^2 \times D^2) \arrow[d, "\cong"] & & \\
D^2 \times S^1 \cup_{S^1 \times S^1} S^1 \times D^2 \arrow[rr, "/\boldsymbol{S^1}"'] & & D^2 \cup_{S^1} D^2 \arrow[uu, "\cong"']
\end{tikzcd}$$

$$\cong \; Z\big(K; (D^2, S^1)\big)$$

Here, T^2 acts on $D^2 \times D^2 \subset \mathbb{C}^2$ and $\boldsymbol{S^1}$ is the diagonal subgroup.

So does the Whitehead product

Consider the attaching map of the top cell of $T^2 \times T^2$

$$\omega\colon S^3 \longrightarrow S^2 \vee S^2$$

$$K = \{\varnothing, v_1, v_2\} \quad \text{and} \quad (X, A) = (D^2, S^1)$$

$$\begin{tikzcd}
S^3 \arrow[rr, "\omega"] \arrow[d, "\cong"] & & S^2 \vee S^2 \\
\partial(D^2 \times D^2) \arrow[d, "\cong"] & & \\
D^2 \times S^1 \cup_{S^1 \times S^1} S^1 \times D^2 \arrow[rr, "c"] & & (S^2 \times *) \cup_{* \times *} (* \times S^2) \arrow[uu, "\simeq"']
\end{tikzcd}$$

$$\cong \; Z\big(K; (D^2, S^1)\big) \qquad\qquad \cong \; Z\big(K; (S^2, *)\big)$$

The map c is induced by the map of pairs $(D^2, S^1) \longrightarrow (S^2, *)$.

A variety of contexts

$(\underline{X}, \underline{A})$	$Z(K; (\underline{X}, \underline{A}))$
(D^2, S^1)	toric geometry and topology
(D^1, S^0)	surfaces, number theory
$(S^1, *)$	robotics, right-angled Artin groups
$(\mathbb{R}\mathrm{P}^\infty, *)$	right-angled Coxeter groups
$(\mathbb{C}, \mathbb{C}^*)$	complements of coordinate arrangements
$(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$	complements of certain non-coordinate arrangements
$(\mathbb{C}\mathrm{P}^\infty, \mathbb{C}\mathrm{P}^{\mathrm{k}})$	monomial ideal rings
(EG, G)	free groups
$(BG, *)$	monodromy, combinatorics
$(PX, \Omega X)$	homotopy theory, Whitehead products
$(S^{2k+1}, *)$	graph products, quadratic algebras

A theorem

The next theorem reduces the complexity of the polyhedral product

$$Z\big(K; (\underline{X}, \underline{A})\big)$$

Theorem 1. Topological spaces $\widehat{Z}(K_I; (\underline{X_I}, \underline{A_I}))\big)$ exist such that

$$\Sigma\big(Z(K; (\underline{X}, \underline{A}))\big) \xrightarrow{\simeq} \Sigma\big(\bigvee_{I \subseteq [m]} \widehat{Z}(K_I; (\underline{X_I}, \underline{A_I}))\big)$$

where $I = (i_1, \cdots, i_k)$ satisfies $1 \leq i_1 < \cdots < i_k \leq m$,

and $K_I \subseteq K$ denotes the full subcomplex of K consisting of all simplices which have their vertices in I.

The content of this theorem lies in the fact that the spaces $\widehat{Z}(K_I; (\underline{X_I}, \underline{A_I}))\big)$ are generally much simpler geometrically than are $Z(K; (\underline{X}, \underline{A}))$.

The polyhedral smash product $\widehat{Z}(L; (\underline{Y}, \underline{B}))\big)$ is defined using smash products in a way entirely analogous to the polyhedral product.

The special case of wedge pairs

A family of CW pairs $(\underline{U}, \underline{V})$ of the form

$$(\underline{U}, \underline{V}) = (\underline{B \vee C}, \underline{B \vee E})$$

with

$$(U_i, V_i) = (B_i \vee C_i, B_i \vee E_i)$$

for all i, and such that

$$E_i \hookrightarrow C_i$$

is a null homotopic inclusion, is called wedge decomposable.

Remark. It's not a surprise that the polyhedral smash product loves pairs of this form because the smash product distributes over the wedge.

The special case of wedge pairs
– a Cartan-type formula –

Wedge decomposable pairs allow for a nice description of the the polyhedral smash product.

Theorem 2. There is a homotopy equivalence

$$\widehat{Z}\big(K;(\underline{B\vee C},\underline{B\vee E})\big)\longrightarrow\bigvee_{I\leq[m]}\widehat{Z}\big(K_I;(\underline{C},\underline{E})\big)\wedge\widehat{Z}\big(K_{[m]-I};(\underline{B},\underline{B})\big)$$

with the convention that

$$\widehat{Z}\big(K_\varnothing;(\underline{B},\underline{B})_\varnothing\big),\ \widehat{Z}\big(K_\varnothing;(\underline{C},\underline{E})_\varnothing\big)\ \text{ and }\ \widehat{Z}\big(K_I;(\varnothing,\varnothing)_I\big)$$

are all S^0.

The equivalence is natural with respect to maps of wedge decomposable pairs.

The wedge lemma helps us

The wedge lemma identifies $\widehat{Z}\big(K_I;(\underline{C},\underline{E})\big)$
because $E_i \hookrightarrow C_i$ is null homotopic.

Corollary 1. There is a homotopy equivalence

$$\widehat{Z}\big(K;(\underline{B\vee C},\underline{B\vee E})\big) \xrightarrow{\ \simeq\ }$$

$$\bigvee_{I\leq[m]}\Big[\Big(\bigvee_{\sigma\in K_I}\Sigma|lk_\sigma(K_I)|\wedge\widehat{D}^I_{\underline{C},\underline{E}}(\sigma)\Big)\wedge\underbrace{\widehat{Z}\big(K_{[m]-I};(\underline{B},\underline{B})_{[m]-I}\big)}_{\text{a smash product of the } B_i}\Big]$$

where

$$\widehat{D}^I_{\underline{C},\underline{E}}(\sigma)=\bigwedge_{j=1}^{m}W_{i_j},\quad\text{with}\quad W_{i_j}=\begin{cases}C_{i_j} & \text{if } i_j\in\sigma\\ E_{i_j} & \text{if } i_j\in I-\sigma.\end{cases}$$

Extending from $(\underline{B \vee C}, \underline{B \vee E})$ to $(\underline{X}, \underline{A})$

We wish to use symmetric products to prove the following:

Theorem 3. Let K have m vertices and suppose that $(\underline{X}, \underline{A})$ is a family of pointed, path-connected pairs of finite CW-complexes.

Then there exist spaces

$$B_j,\ C_j,\ E_j, \quad 1 \leq j \leq m$$

which are finite wedges of spheres and mod-n Moore spaces so that

$$H_*(\widehat{Z}(K; (\underline{X}, \underline{A}))) \ \cong\ H_*(\widehat{Z}(K; (\underline{B \vee C}, \underline{B \vee E})))$$

over the integers.

Other computations: BBCG 2014, Qibing Zheng 2014

Symmetric products

Definition. Let $(X, *)$ denote a pointed topological space.

The *m-fold symmetric product* for $(X, *)$ is the orbit space

$$SP^m(X) = X^m/\Sigma_m$$

where the symmetric group on m-letters Σ_m acts on the left by permutation of coordinates.

There are natural maps

$$\begin{array}{rccc} \boldsymbol{e}\colon & SP^m(X) & \longrightarrow & SP^{m+1}(X) \\ & [x_1, x_2, \ldots, x_m] & \mapsto & [x_1, x_2, \ldots, x_m, *] \end{array}$$

which allow for the definition of the *infinite* symmetric product as a colimit

$$SP(X) = \operatorname*{\mathbf{colim}}_{1 \leq m} SP^m(X).$$

The next theorem is a version of a result due to Dold & Thom.

Theorem 3. Given a pointed, path-connected pair of finite CW-complexes $(X, A, *)$, then

(i) $SP(X)$ is homotopy equivalent to a product of Eilenberg-MacLane spaces

$$\prod_{1\leq q\leq\infty} K(H_q(X), q)$$

$$\Longrightarrow \quad \pi_i SP(X) \;\cong\; \widetilde{H}_i(X).$$

(ii) When A is a closed subcomplex of X, the natural map

$$SP(X) \longrightarrow SP(X/A)$$

is a quasi-fibration with quasi-fibre $SP(A)$.

Symmetric products

A natural map

$$SP^{q_1}(X_1) \wedge SP^{q_2}(X_2) \wedge \cdots \wedge SP^{q_m}(X_m)$$
$$\xrightarrow{\widehat{\boldsymbol{\theta}}} SP^q(X_1 \wedge X_2 \wedge \cdots \wedge X_m),$$

is constructed for $q = q_1 q_2 \cdots q_m$ by setting

$$\widehat{\boldsymbol{\theta}}\Big(\big[[x_{11}, x_{12}, \ldots, x_{1q_1}], [x_{21}, x_{22}, \ldots, x_{2q_2}], \ldots, [x_{m1}, x_{m2}, \ldots, x_{mq_m}]\big]^{\wedge}\Big)$$
$$= \Big[\prod_{\substack{1 \le j_t \le q_t \\ 1 \le t \le m}} \big[x_{1j_1}, x_{2j_2}, \ldots, x_{mj_m}\big]^{\wedge}\Big]$$

where square brackets [ ] are used to denote equivalence classes in the symmetric product, and $[\ ]^{\wedge}$ for the smash products.

Symmetric products and polyhedral products

The map $\widehat{\boldsymbol{\theta}}$ extends in a natural way to give a map of colimits

$$SP(X_1) \wedge SP(X_2) \wedge \cdots \wedge SP(X_m) \xrightarrow{\widehat{\boldsymbol{\theta}}} SP(X_1 \wedge X_2 \wedge \cdots \wedge X_m)$$

This induces a structure map,

$$\boldsymbol{\zeta}\colon\ \widehat{Z}\big(K; (\underline{SP(X)}, \underline{SP(A)})\big) \longrightarrow SP\big(\widehat{Z}\big(K; (\underline{X}, \underline{A})\big)\big).$$

in not an entirely obvious way.

A definition (which will not restrict us)

The pairs

$$(\underline{U}, \underline{V}) \text{ and } (\underline{X}, \underline{A})$$

are said to have strongly isomorphic homology provided three things happen:

(1) There are isomorphisms of singular homology groups

$$\alpha_j : H_*(U_j) \to H_*(X_j)$$

and

$$\beta_j : H_*(V_j) \to H_*(A_j)$$

(2) There is a commutative diagram

$$\begin{array}{ccc} \bar{H}_i(V_j) & \xrightarrow{\lambda_{j*}} & \bar{H}_i(U_j) \\ {\scriptstyle \beta_j}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_j} \\ \bar{H}_i(A_j) & \xrightarrow{\iota_{j*}} & \bar{H}_i(X_j), \end{array}$$

where $\lambda_j : V_j \subset U_j$, and $\iota_j : A_j \subset X_j$ are the natural inclusions.

A definition

(3) There is an induced morphism of exact sequences for which all vertical arrows are isomorphisms:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & ker(\lambda_{j_*}) & \longrightarrow & \bar{H}_i(V_j) & \xrightarrow{\lambda_{j_*}} & \bar{H}_i(U_j) & \longrightarrow & coker(\lambda_{j_*}) & \longrightarrow & 0 \\
\Big\downarrow & & \beta_j\Big\downarrow & & \beta_j\Big\downarrow & & \Big\downarrow\alpha_j & & \Big\downarrow\bar{\alpha}_j & & \Big\downarrow \\
0 & \longrightarrow & ker(\iota_{j_*}) & \longrightarrow & \bar{H}_i(A_j) & \xrightarrow{\iota_{j*}} & \bar{H}_i(X_j) & \longrightarrow & coker(\iota_{j_*}) & \longrightarrow & 0
\end{array}$$

where $\bar{\alpha}_j$ is induced by α_j.

Strongly isomorphic pairs are good because they ensure that everything behaves well with respect to the Künneth theorem.

A consequence of the definition

Lemma. Let $(\underline{X}, \underline{A})$, and $(\underline{U}, \underline{V})$ be pairs of pointed, path connected finite CW complexes.

Then, if they have strongly isomorphic homology groups, there is an isomorphism of singular homology groups

$$\bar{H}_*\big(\widehat{D}_{(\underline{X},\underline{A})}(\sigma)\big) \longrightarrow \bar{H}_*\big(\widehat{D}_{(\underline{U},\underline{V})}(\sigma)\big).$$

for any face σ in the simplicial complex K,

Back to wedge decomposable pairs

Lemma. Let $(\underline{X}, \underline{A})$ consist of pointed, path-connected pairs of finite CW-complexes.

Then there exist wedges of spheres, and mod-p^r Moore spaces

$$(\underline{B \vee C}, \underline{B \vee E})$$

together with isomorphisms of singular homology groups

$$\alpha_j \colon H_*(B_j \vee C_j) \to H_*(X_j)$$

and

$$\beta_j \colon H_*(B_j \vee E_j) \to H_*(A_j),$$

which give strong homology isomorphisms.

The inclusions $E_j \to C_j$ are null-homotopic, so the pairs

$$(\underline{B \vee C}, \underline{B \vee E})$$

satisfy condition of wedge decomposability.

Getting the most out of a homology isomorphism

Suppose that $(\underline{U}, \underline{V})$ and $(\underline{X}, \underline{A})$ are pointed, connected, pairs of finite CW-complexes, with strongly isomorphic homology groups.

[We have in mind: $(\underline{U}, \underline{V}) = (\underline{B \vee C}, \underline{B \vee E})$.]

Then a multiplicative map of pairs

$$g : \big(SP(U), SP(V)\big) \longrightarrow \big(SP(X), SP(A)\big)$$

exists inducing strongly isomorphic homology groups including a commutative diagram

$$\begin{array}{ccc}
\bar{H}_i\big(SP(V_j)\big) & \xrightarrow{\lambda_*} & \bar{H}_i\big(SP(U_j)\big) \\
{\scriptstyle g_*}\big\downarrow & & \big\downarrow{\scriptstyle g_*} \\
\bar{H}_i\big(SP(A_j)\big) & \xrightarrow{\iota_*} & \bar{H}_i\big(SP(X_j)\big)
\end{array}$$

Getting the most out of a homology isomorphism

...and another for which all vertical arrows are isomorphisms:

$$\begin{tikzcd} ker(\lambda_*) \arrow[r] \arrow[d, "g_*"'] & \bar{H}_i(SP(V_j)) \arrow[r, "\lambda_*"] \arrow[d, "g_*"'] & \bar{H}_i(SP(U_j)) \arrow[r] \arrow[d, "g_*"] & coker(\lambda_*) \arrow[d, "\bar{g}_*"] \\ ker(\iota_*) \arrow[r] & \bar{H}_i(SP(A_j)) \arrow[r, "\iota_*"] & \bar{H}_i(SP(X_j)) \arrow[r] & coker(\iota_*) \end{tikzcd}$$

where $\bar{g}_*$ is induced by g_*.

Remark. The map g might not be homotopic to the one which is given automatically by virtue of the fact that $SP(X)$ is homotopy equivalent to a product of Eilenberg-MacLane spaces.

Getting the most out of a homology isomorphism

Applying the functor $\widehat{D}_{(-,-)}(\sigma)$ to the map of pointed pairs

$$g\colon \big(\underline{SP(B\vee C)}, \underline{SP(B\vee E)}\big) \longrightarrow (\underline{SP(X)}, \underline{SP(A)})$$

(which induces a strong isomorphism in homology),

we get a morphism

$$\widehat{D}_{(\underline{SP(B\vee C)},\,\underline{SP(B\vee E)})}(\sigma) \xrightarrow[\simeq]{\widehat{D}(\sigma;g)} \widehat{D}_{(\underline{SP(X)},\underline{SP(A)})}(\sigma)$$

and, for each $\tau \subset \sigma$, a commutative diagram

$$\begin{tikzcd}
\widehat{D}_{(\underline{SP(B\vee C)},\,\underline{SP(B\vee E)})}(\tau) \arrow[r, "\widehat{D}(\tau;g)", "\simeq"'] \arrow[d, "\beta"] & \widehat{D}_{(\underline{SP(X)},\underline{SP(A)})}(\tau) \arrow[d, "\beta"] \\
\widehat{D}_{(\underline{SP(B\vee C)},\,\underline{SP(B\vee E)})}(\sigma) \arrow[r, "\widehat{D}(\sigma;g)", "\simeq"'] & \widehat{D}_{(\underline{SP(X)},\underline{SP(A)})}(\sigma)
\end{tikzcd}$$

Getting the most out of a homology isomorphism

Further there are induced morphisms of commutative diagrams via the structure map ζ.

$$\begin{array}{ccc}
\widehat{D}_{(\underline{SP(B\vee C)},\ \underline{SP(B\vee E)})}(\sigma) & \xrightarrow[\simeq]{} & \widehat{D}_{(\underline{SP(X)},\underline{SP(A)})}(\sigma) \\
\Big\downarrow \zeta & & \Big\downarrow \zeta \\
SP\big(\widehat{D}_{(\underline{B\vee C},\ \underline{B\vee E})}(\sigma)\big) & \xrightarrow[\simeq]{} & SP\big(\widehat{D}_{(\underline{X},\ \underline{A})}(\sigma)\big)
\end{array}$$

where the lower horizontal arrow is a homotopy equivalence by the Dold-Thom theorem.

A conclusion

So, there is a map

$$\begin{tikzcd}
SP\big(\bigcup_{\sigma\in K}\widehat{D}_{(\underline{B\vee C},\,\underline{B\vee E})}(\sigma)\big) \arrow[r, "\mu"] \arrow[d, "="] & SP\big(\bigcup_{\sigma\in K}\widehat{D}_{(\underline{X},\,\underline{A})}(\sigma)\big) \arrow[d, "="] \\
SP\big(\widehat{Z}(K;(\underline{B\vee C},\ \underline{B\vee E}))\big) \arrow[r, "\mu"] & SP\big(\widehat{Z}(K;(\underline{X},\ \underline{A}))\big)
\end{tikzcd}$$

Finally, we invoke Quillen's Projection Lemma to conclude the proof.

Projection Lemma

Lemma. Let $\mathcal{D}$ and $\mathcal{E}$ be finite diagrams of finite CW complexes over the same finite category $\mathfrak{C}$, satisfying:

(i) All inclusions in the intersection poset are closed cofibrations.

(ii) We have colimits

$$U = \bigcup_{\alpha \in \mathfrak{C}} D_\alpha \quad \text{and} \quad X = \bigcup_{\alpha \in \mathcal{C}} E_\alpha$$

(iii) There is a map

$$\mu\colon SP(U) \longrightarrow SP(X)$$

which restricts to homotopy equivalences on

$$\mu|_{SP(D_\alpha)} : SP(D_\alpha) \longrightarrow SP(E_\alpha)$$

Then μ is a homotopy equivalence.

An example with numbers

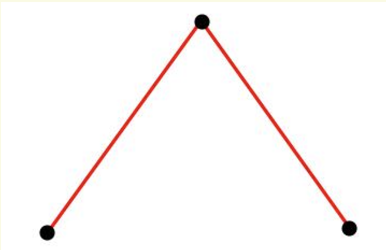
Consider the composite

$$f\colon \mathbb{C}P^2 \hookrightarrow \mathbb{C}P^3 \to \mathbb{C}P^3/\mathbb{C}P^1.$$

and denote the mapping cylinder of f by M_f.

We shall describe describe the Poincaré series of $\widehat{Z}\big(K;(M_f,\mathbb{C}P^2)\big)$ for any for the special case

$$K = \big\{\{v_1\},,\{v_2\},\{v_3\},\{v_1,v_2\},\{v_1,v_3\}\big\}.$$



An example with numbers

For $(X,A) = (M_f, \mathbb{C}P^2)$, we have

$$(U,V) = \Big(S^4 \vee S^6,\ S^4 \vee S^2\Big).$$

so that

$$B = S^4,\ C = S^6 \text{ and } E = S^2$$

The theorem gives:

$$\widetilde{H}^*\big(\widehat{Z}\big(K;(M_f,\mathbb{C}P^2)\big)\big) \cong \widetilde{H}^*\big(\widehat{Z}(K;(B \vee C, B \vee E))\big).$$

An example with numbers

Applying the Cartan decomposition we get:

$$\widehat{Z}(K;(B\vee C, B\vee E)) \xrightarrow{\simeq} \bigvee_{I\leq[m]} \widehat{Z}\big(K_I;\big(S^6,\ S^2\big)\big) \wedge \widehat{Z}\big(K_{[m]-I};(S^4,\ S^4)\big)$$
$$= \bigvee_{I\leq[m]} \widehat{Z}\big(K_I;\big(S^6,\ S^2\big)\big)\ \wedge\ \big(S^4\big)^{\wedge|[m]-I||}$$

The Wedge Lemma decomposes

$$\widehat{Z}\big(K_I;\big(S^6,\ S^2\big)\big)$$

further by enumerating all the links $|lk_\sigma(K_I)|$. The reduced Hilbert-Poincaré series for $\widehat{Z}(K;(B\vee C, B\vee E))$, and hence for $\widehat{Z}(K;(\underline{X},\underline{A}))$ is

$$\overline{P}\big(\widehat{Z}(K;(B\vee C, B\vee E)), t\big)$$
$$= \sum_{I\leq[m]}\Big[\sum_{\sigma\in K_I}\big[t\cdot\overline{P}(|\mathrm{lk}_\sigma(K_I)|, t)\cdot\overline{P}\big(\widehat{D}^I_{\underline{S^6},\underline{S^2}}(\sigma), t\big)\big]\cdot\prod_{j\in[m]-I}\overline{P}(B_j, t)\Big]$$

An example with numbers

The cohomology of $(M_f, \mathbb{C}P^2)$ satisfies

$$H^*(M_f) = \mathbb{Z}\{b_4, c_6\} \quad \text{and} \quad H^*(\mathbb{C}P^2)) = \mathbb{Z}\{e_2, b_4\}$$

where the dimensions of the classes are given by the subscripts.

The classes $\{e_2, b_4, c_6\}$ supported on the vertex i are denoted by $\{e_2^i, b_4^i, c_6^i\}$.

We illustrate the computation by determining the summand corresponding to

$$I = \{2, 3\} \quad \text{and} \quad \sigma = \varnothing.$$

An example with numbers

In this case, $I = \{2,3\}$ and $\sigma = \varnothing$. we have:

(i) $\widehat{D}^I_{\underline{C},\underline{E}}(\sigma) = \; E_2 \wedge E_3 = S^2 \wedge S^2$ and $\widetilde{H}\big(\widehat{D}^I_{\underline{C},\underline{E}}(\sigma)\big) = k\{e_2^2 \otimes e_2^3\}$

and so we get $\overline{P}\big(\widehat{D}^I_{\underline{S^6},\underline{S^2}}(\sigma), t\big) = t^4$.

(ii) Next, since $[m] - I = \{1\}$, we have:

$$\prod_{j\in\{1\}} \overline{P}(B_j, t) = \overline{P}(B_1, t) = \overline{P}(S^4, t) \implies \overline{P}(b_4^1, t) = t^4.$$

(iii) Turning to the links,

$$|lk_{\varnothing}(K_I)| \;=\; |\{\{2\},\{3\}\}| = S^0$$

so that $t \cdot \overline{P}\big(|lk_{\varnothing}(K_I), t|\big) \;=\; t.$

An example with numbers

Finally, for the case at hand, we get a contribution of t^9 to the Poincaré series for $H^*\big(\widehat{Z}(K;(\underline{X},\underline{A}))\big)$.

Continuing in this way, we arrive at the (reduced) Poincaré series:

$$\overline{P}\big(H^*(\widehat{Z}(K;(M_f,\mathbb{C}P^2)),t\big) = t^9 + t^{11} + 3t^{12} + 5t^{14} + 2t^{16}$$