Today: global Torelli Theorem for complex K3 surfaces.

(Rapoport, Burns 1975)

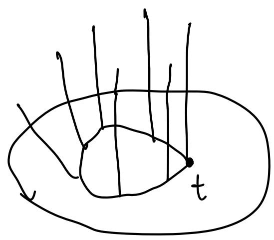
First we discuss monodromy group of complex K3.

Fix a complex K3 surface S.

Consider a family $\begin{matrix} \mathcal{S} & S \\ \downarrow & \downarrow \\ B & \ni t \end{matrix}$ of K3 surfaces,

smooth, proper.

have a group homomorphism $\pi_1(B, t) \longrightarrow \mathrm{Aut}(H^2(S, \mathbb{Z}))$



B

a loop based on t $\rightsquigarrow$ homeomorphism from S to S

smooth loop. diffeomorphism.

this induce an isomorphism $H^2(S, \mathbb{Z}) \xrightarrow{\cong} H^2(S, \mathbb{Z})$

as K3 lattice.

the induced isometry of $H^2(S, \mathbb{Z})$ does not depend on the homotopy class of the loop.

So we have: $\pi_1(B, t) \longrightarrow \mathrm{Aut}(H^2(S, \mathbb{Z}))$.

this homomorphism is called the monodromy representation of

$\pi_1(B,t)$, and the image is called the monodromy group of $\begin{matrix} S \\ \downarrow \\ B \end{matrix}$.

All possible such groups generated a subgroup $\mathrm{Mon}(S) \subset \mathrm{Aut}(H^2(S,\mathbb{Z}))$

We call $\mathrm{Mon}(S)$ the monodromy group of S.

Thm: $\mathrm{Mon}(S) = \mathrm{Aut}^+(H^2(S,\mathbb{Z})) \subset \mathrm{Aut}(H^2(S,\mathbb{Z}))$.

Here, $\mathrm{Aut}^+(H^2(S,\mathbb{Z}))$ is the group of isometries of $H^2(S,\mathbb{Z})$ preserving the orientation of a positive 3-subspace of $H^2(S,\mathbb{R})$.

Spinor norm: $\delta \in (\Lambda_{K3})_{\mathbb{R}}$, $\delta^2 \neq 0$,

can define reflection $s_\delta : v \longmapsto v - 2\frac{(v,\delta)}{\delta^2} \cdot \delta$.

$[\, s_\delta(\delta) = -\delta, \quad s_\delta(v) = -v \text{ if } v \perp \delta \,]$.

define the spinor norm of s_δ to be $\begin{cases} -1 & \text{if } \delta^2 > 0 \\ 1 & \text{if } \delta^2 < 0. \end{cases}$

Thm (Cartan - Dieudonné): any isometry of $(\Lambda_{K3})_{\mathbb{R}}$ can be written as a composition of finitely many reflection.

$\Rightarrow$ we have a well-defined group homomorphism

$\mathrm{Aut}((\Lambda_{K3})_{\mathbb{R}}) \longrightarrow \{\pm 1\}$.

such that $s_\delta \longmapsto \begin{cases} -1 & \delta^2 > 0 \\ 1 & \delta^2 < 0. \end{cases}$

the image of an isometry in $\{\pm 1\}$ is called its spinor norm.

Rmk: An isometry of $H^2(S,\mathbb{R})$ has spinor norm 1 if and only if it preserves the orientation of certain positive 3-subspace.

$\delta^2 > 0$, take a positive 3-subspace W containing δ.

$W = W_0 \oplus \mathbb{R}\delta$, $\dim_{\mathbb{R}} W_0 = 2$.

$S_\delta : \delta \mapsto -\delta$, fix elements in W_0.

$S_\delta : W \longrightarrow W$, change orientation.

[So the spinor norm of S_δ should be -1]

$\delta^2 < 0$, take $\underset{\text{positive 3-subspace}}{W \subset \delta^\perp}$,

then S_δ fixes W, hence fixes the orientation.

So S_δ should have spinor norm 1.

Proof of the Thm ($\mathrm{Mon}(S) = \mathrm{Aut}^+(H^2(S,\mathbb{Z}))$):

$\mathrm{Aut}^+(H^2(S,\mathbb{Z}))$ is the kernel of $\mathrm{Aut}(H^2(S,\mathbb{Z})) \to \{\pm 1\}$.

has index two in $\mathrm{Aut}(H^2(S,\mathbb{Z}))$.

First we show: for any $\delta \in H^2(S,\mathbb{Z})$ with $\delta^2 = -2$,

we have $S_\delta \in \mathrm{Mon}(S)$.

$S = (M, I)$, $\delta \in H^2(M,\mathbb{Z})$.

deform I to I', (M, I') is a new K3 surface such that

$\mathrm{Pic}(M, I') = \mathbb{Z}\delta$ (need to use the surjectivity of $\wp: N \to D$)

[let $H^{2,0}(M, I')$ be a generic element in $\delta^{\perp}$ in D_{K3}]

δ or $-\delta$ is represented by a $\mathbb{P}^1$ on (M, I').

Riemann-Roch

$H^0(\delta) - H^1(\delta) + H^2(\delta)$

$= \chi(\mathcal{O}) + \frac{1}{2}\delta^2 = 1$

$\Rightarrow$ $H^0(\delta)$ or $H^2(\delta) = H^0(-\delta)$ is non-empty

WLOG, $\delta = [C]$, $C \cong \mathbb{P}^1$.

$\exists$ $(M, I') \to X$, analytic morphism to a complex surface X with one nodal point.

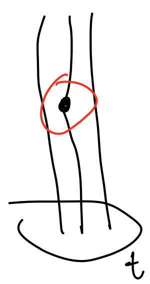
here C is contracted to the node,

Locally near the node, X is defined by equation $x^2 + y^2 + z^2 = 0$

X can fit into a proper flat family $\begin{matrix}\mathcal{X}\\ \downarrow \\ \Delta \ni 0\end{matrix}$ with $\mathcal{X}_0 = X$

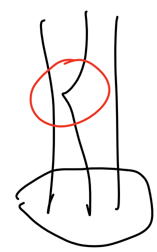
and $\begin{matrix}\mathcal{X} - X\\ \downarrow \\ \Delta - 0\end{matrix}$ is smooth.

fiber of $\begin{matrix}\mathcal{X} - X\\ \downarrow \\ \Delta - 0\end{matrix}$ are smooth K3 surface with underlying manifold M



locally $x^2 + y^2 + z^2 = t$

t



$x^2 + y^2 + z^2 = u^2$

Δ

u

$t = u^2$

$\mathcal{X} - X \longleftarrow (\mathcal{X} - X) \times_{\Delta^*} \Delta^*$ $\quad\quad \Delta^* = \Delta - 0$

$\downarrow \quad \square \quad \downarrow$

$$\begin{tikzcd} \Delta^* & \Delta^* \arrow[l, "1:2"'] \end{tikzcd}$$

$(\mathcal{X}-X)\times_{\Delta^*}\Delta^*$ can be completed into a smooth proper family of K3 surfaces over Δ.

$\Rightarrow$ the monodromy for $\begin{matrix}\mathcal{X}-X\\ \downarrow \\ \Delta^*\end{matrix}$ is a reflection in δ.

[Picard-Lefschetz formula].

So $s_\delta \in \mathrm{Mon}(S)$.

Wall, Ebeling, Kneser: any element in $\mathrm{Aut}^+(H^2(S,\mathbb{Z}))$ can be written as a product of finitely many reflections in (-2)-classes.

$\Rightarrow$ $\mathrm{Mon}(S) \supset \mathrm{Aut}^+(H^2(S,\mathbb{Z}))$.

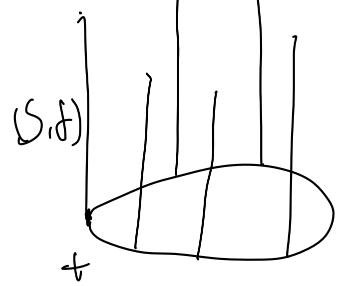
So either $\mathrm{Mon}(S) = \mathrm{Aut}^+$ or $\mathrm{Mon}(S) = \mathrm{Aut}$

Claim: $-\mathrm{id} \notin \mathrm{Mon}(S)$.

otherwise, there exists a family $\begin{matrix}\mathcal{S} & S\\ \downarrow & \downarrow \\ B & \ni t\end{matrix}$, a smooth loop based on t, giving rise $-\mathrm{id}$.

take a marking $f: H^2(S,\mathbb{Z}) \to \Lambda_{K3}$.

extend the marking along the loop:



when go back to t, f is changed to $-f$.

So (S,f), $(S,-f)$ as points in N, lie in one connected component,

Note that (S,f) and $(S,-f)$ has same period in D.

$[\ f(H^{2,0}(S)) = -f(H^{2,0}(S))\ ]$.

By isomorphism between connected component of $\bar{N}$ and D, we know (S,f) and $(S,-f)$ represent the same point in $\bar{N}$

Note that (S,f) and $(S,-f)$ represent different points in N.
otherwise, $\exists\ S \xrightarrow{\sim} S$ induces $-id$ on $H^2(S,\mathbb{Z})$.
But Kähler cone is sent to Kähler cone, impossible.

So (S,f), $(S,-f)$ are different but inseparable points in N,

[Lem: If (S_1,f_1), (S_2,f_2) are different, inseparable pts in N, then S_1, S_2 has non zero Pic].

If S is chosen generically at beginning, the we already see contradiction $\Rightarrow$ $-id \notin Mon(S)$.

The statement $-\mathrm{id} \notin \mathrm{Mon}(S)$ does not depend on the choice of S

So for any S, $-\mathrm{id} \notin \mathrm{Mon}(S)$.

$$\Rightarrow \quad \mathrm{Mon}(S) = \mathrm{Aut}^+(H^2(S,\mathbb{Z})).$$

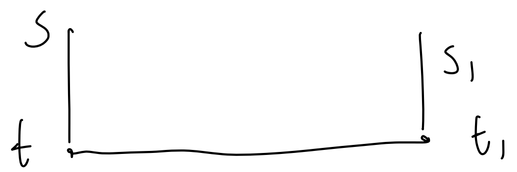
Corollary: N and $\bar{N}$ have two connected components.

Pf: Fix S, a marking f,

(S,f), $(S,-f)$ lie in different connected components of N,

take $(S_1,f_1) \in N$,

take a family
$$\begin{array}{ccccc} \mathcal{S} & & S & & S_1 \\ \downarrow & & \downarrow & & \downarrow \\ B & \ni & t & & t_1 \end{array}$$



extend f along the path,

get marking f_2 on S_1.

f_1, f_2 two markings of S_1,

$f_1^{-1} \circ f_2 \in \mathrm{Aut}(H^2(S_1,\mathbb{Z}))$

either $\underset{\mathrm{Mon}(S_1)}{f_1^{-1} \circ f_2 \in \mathrm{Aut}^+}$, or $\underset{\mathrm{Mon}(S_1)}{-f_1^{-1} \circ f_2 \in \mathrm{Aut}^+}$.

$$\mathrm{Aut}^+ = \mathrm{Mon}(S_1)$$

If $f_1^{-1} \circ f_2 \in \mathrm{Mon}(S_1) \Rightarrow (S,f), (S_1,f_1)$ are connected in N

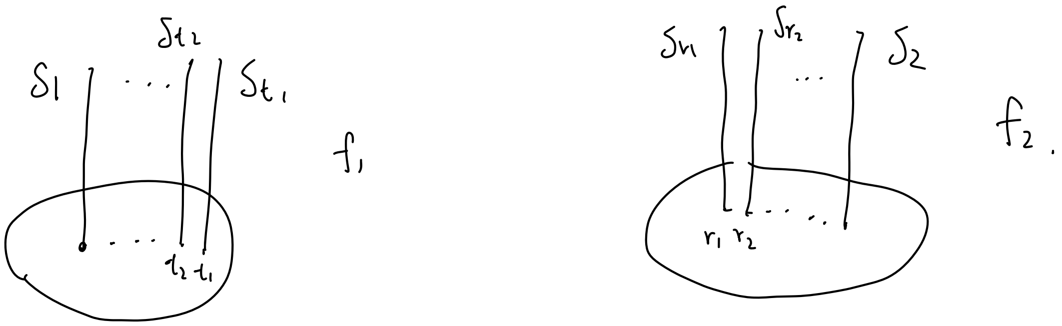
If $-f_1^{-1} \circ f_2 \in \mathrm{Mon}(S_1) \Rightarrow (S,-f), (S_1,f_1)$ are connected.

Lem: (S_1, f_1), $(S_2, f_2) \in N$ distinct, inseparable,

then $S_1 \cong S_2$ and $\wp(S_1, f_1) = \wp(S_2, f_2)$ is contained

in $\alpha^\perp$ for some $0 \neq \alpha \in \Lambda_{K3}$.

Pf: (S_1, f_1) and (S_2, f_2) are inseparable,



(S_{t_i}, f_1) and (S_{r_i}, f_2) represent the same point in N

$\exists\ \psi_i : S_{t_i} \xrightarrow{\cong} S_{r_i}$, with:

$$\begin{tikzcd} H^2(S_{t_i}, \mathbb{Z}) \arrow[r, "f_1"] & \Lambda_{K3} \\ H^2(S_{r_i}, \mathbb{Z}) \arrow[u, "\psi_i^*"] \arrow[ur, "f_2"'] \arrow[ur, phantom, "\circlearrowleft"] & \end{tikzcd}$$

graph of ψ_i: $\Gamma_i = \{ (x, \psi_i(x)) \mid x \in S_{t_i} \} \subset S_{t_i} \times S_{r_i}$.

Let $i \to \infty$, then $\Gamma_i \to \Gamma_\infty \subset S_1 \times S_2$.

here Γ_∞ is an analytic cycle in $S_1 \times S_2$

[Rapoport - Burns]. see also [Looijenga - Peters 1980

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Torelli theorems for Kähler K3 surfaces]

Bishop 1964 ; to show Γ_∞ has analytic structure.

need to show Γ_i has finite volume.

$$\Gamma_\infty = Z + \sum_{i=0}^{k} (C_i \times C_i')$$.

Z is a graph for an isom $S_1 \xrightarrow{\sim} S_2$

$C_i \subset S_1$, $C_i' \subset S_2$ have dimension 1

$[\Gamma_\infty] : H^2(S_1, \mathbb{Z}) \xrightarrow{\sim} H^2(S_2, \mathbb{Z})$.

is equal to $f_2^{-1} \circ f_1$

If no C_i, C_i', then $[Z] = f_2^{-1} \circ f_1$.

$\Rightarrow$ $f_2^{-1} \circ f_1$ is induced by an isom between S_1, S_2.

$\Rightarrow$ (S_1, f_1), (S_2, f_2) represent the same point in N,

contradiction.

So C_i, C_i' exist.

$\Rightarrow$ $P_{ic}(S_1)$, $P_{ic}(S_2)$ non zero. $\square$