Lecture 19   18-May 2022

Recall: $V \subset R \subset BD^*$

$BD \hookrightarrow BD^{**} \Rightarrow$ elements of BD are functions on R.

$\hat{p} \in R \Rightarrow$ For any $f \in BD$

$\Rightarrow \hat{p}(f) \in \mathbb{R}$

$\Rightarrow \hat{f}(\hat{p}) = \hat{p}(f)$ where $\hat{f} \in (BD^*)^*$

<u>Def</u> (Γ, r) network, p one-sided infinite path in Γ.

$V(p)$ = set of vertices in $p \subset R$

$\overline{V(p)}$ closure in R

extreme points $E(p) = \overline{V(p)} \cap bR$

(Recall: $bR = R - V$)

Thm (Γ, r) infinite transient network. p one-sided infinite path.

$$p = x_0 \sim x_1 \sim x_2 \sim \cdots$$

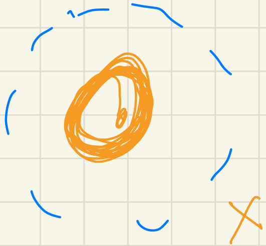
s.t. $\displaystyle\sum_{n=0}^{\infty} r(x_n, x_{n+1}) < \infty$

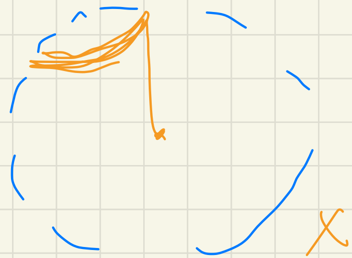
Claim: $E(p) \subset \overset{\Delta + (bR - \Delta)}{\overset{\shortparallel}{bR}}$

Then $E(p)$ is a singleton

and $E(p) \subset \Delta$

Cases:





$\left(C_{xy} = \frac{1}{r_{xy}} \right)$

Pf: Take $S \in BD$. For all $n > m$,

$$|f(x_n) - f(x_m)| = \left| \sum_{k=m}^{n-1} f(x_{k+1}) - f(x_k) \right|$$

$$\le \sum_{k=m}^{n-1} \left(|f(x_{k+1}) - f(x_k)| \sqrt{C_{xy}} \right) \cdot \sqrt{r_{xy}}$$

Cauchy-schwartz

$$\le \sqrt{\sum_{k=m}^{n-1} C_{xy} |f(x_{k+1}) - f(x_k)|^2} \sqrt{\sum_{k=m}^{n-1} r_{xy}}$$

$$\cancel{\le D(S) \quad C}$$

$\Rightarrow$ $f(x_n)$ converges uniquely to some number.

Define $\hat{p}$ function on BD,

Let $f \in BD$, $\hat{p}(f) = \lim_{n \to \infty} f(x_n) \in \mathbb{R}$,

① $\hat{p}$ is linear, i.e. $\hat{p}(af+g) = a\hat{p}(f) + \hat{p}(g)$

② multiplicative since $\hat{p}(fg) = \lim_{n \to \infty} (fg)(x_n) = \lim f(x_n) \lim g(x_n)$

$$= \hat{p}(f)\,\hat{p}(g).$$

③. nontrivial since $\mathbb{1}$ const. function $\in BD$.

$$\hat{p}(\mathbb{1}) = \lim \mathbb{1}(x_n) = 1 \neq 0,$$

$$(\text{Recurrent} \Leftrightarrow \mathbb{1} \in D_0)$$

$$\Rightarrow \quad \hat{p} \in R.$$

Since BD seperate points in R,

$$\Rightarrow \quad \hat{p} \neq x_n \quad \text{for all } n.$$

$$\hat{p} \neq x \quad \text{for all } x \in V.$$

E.g.

$$g_{x_0}(x) := \begin{cases} 1 & \text{for all } x \neq x_0 \\ 0 & \text{for } x = x_0 \end{cases}$$

$\Rightarrow \quad \hat{p} \in R - V = bR.$

Uniquness of limit $\Rightarrow$ $E(p) \subset bR$ is singleton

Claim: $\hat{p} \in \Delta$. i.e. $f(\hat{p}) = 0 \quad \forall f \in BD_0$.

Observation: If f has finite support, then

Recall: $f(\hat{p}) = \lim_{n\to\infty} f(x_n)$

If $f(\hat{p}) \neq 0 \Rightarrow$ the one-sided path p hit $\mathrm{supp}(f)$ infinitly many time

$\Rightarrow$ Contradict $\sum_p r(x_n, x_{n+1}) < \infty$

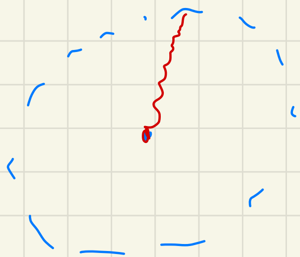
$\Rightarrow \quad f(\hat{p}) = 0.$

Question: Suppose $f \in BD_0$, i.e. $\exists\ f_n \in BD$, with finite support and $\|f_n - f\| \to 0$ as $n \to \infty$

"Regard $\hat{p} \in BD^*$, $\quad \hat{p}(f_n) = 0 \overset{?}{\Rightarrow} \hat{p}(f) = 0$"

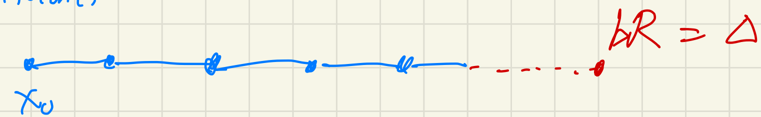
Conclusion, $\quad f(\hat{p}) = 0 \quad \forall f \in BD_0$

$\Rightarrow \quad \hat{p} \in \Delta$ harmonic boundary.



Ex. (Γ, r) transient.

$\mathbb{Z}^+$



Choose r s.t. $\sum_{n=1}^{\infty} r(x_n, x_{n+1}) < \infty$

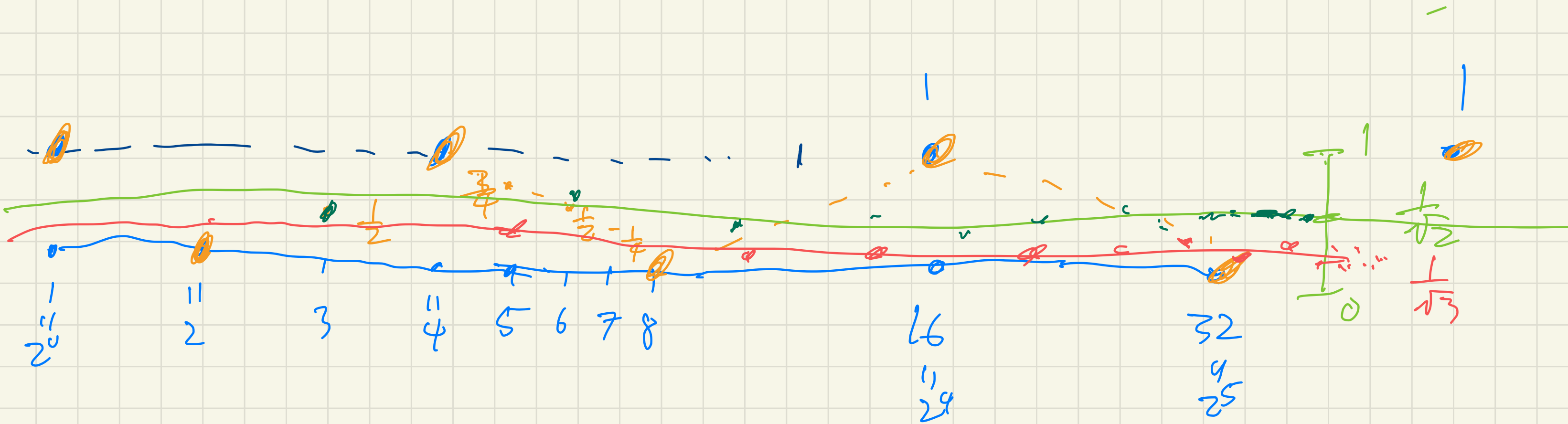
$bR = E(p)$ only one-point

Ex (Γ, r), $\Gamma = \mathbb{Z}^+$, $r \equiv 1$ recurrent.

Claim: $\Delta = \phi$, bR uncountable set.

Observation: $\Delta = \phi$ "$\sim$" no one-sided infinite path with $\sum_{n=1}^{\infty} r(x_n, x_{n+1}) < \infty$

Construct $f : \mathbb{Z}_{>0} \to \mathbb{R}$ as follows



Check: f is bounded and has finite energy

$\Rightarrow \quad f \in BD$

$f(V) \subset [0,1]$ is a dense subset.

$$\Rightarrow \quad f(R) = \overline{f(V)} = [0,1].$$

Recall: R is compact and V is dense in R,

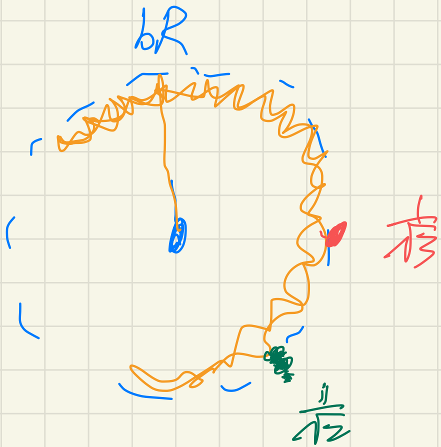
Note: Take $y \in [0,1] - f(V)$

$$\Rightarrow \exists\, p \in (\underset{\substack{\parallel \\ bR}}{R - V}) \quad \text{s.t.} \quad f(p) = y$$

On the other hand, $f(V)$ is countably infinite.

$\Rightarrow$ $[0,1] - f(V)$ is uncountably infinite

$\Rightarrow$ bR is uncountably infinite. $\#$



$$bR = \underset{\emptyset}{\underset{\|}{\Delta'}} \cup \underset{\text{uncountably many.}}{\underset{\|}{bR - \Delta}}$$

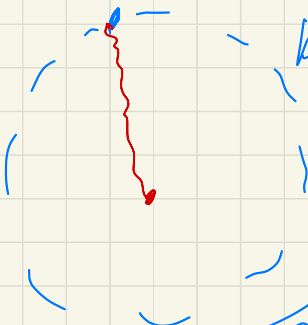
Ex. Γ locally finite graph of bounded degree satisfying a strong isoperimetric inequality (will study later on).

(Some special transient graph).

and $r \geq 1$,

Then, $bR = \Delta = \overline{\{ \bigcup E(p) \mid p \in \mathcal{P} \}}$

where $\mathcal{P}$ = set of all one-sided infinite path.



$bR = \Delta$

<u>Rmk</u> Generally, it is difficult to describe bR explicitly.

Some statements about Δ without proofs.

Thm (Γ, r) transient. If $u \in HBD$ and non-trivial s.t. $u \geq 0$ on Δ

$\Rightarrow$ $u > 0$ on V

Thm $BD_0 = \{ f \in BD \mid f(x) = 0 \quad \forall x \in D \}$.

Thm Every $f \in D$ has continuous extension to R which takes value in $[-\infty, \infty]$.

PS. Assume $f \geq 0$.

$\exists$ $f_n \in BD$ s.t. $|f - f_n| \to 0$ as $n \to \infty$

and $f \geq f_{n+1} \geq f_n$

Pick $p \in R$, $f(p) := \lim_{n\to\infty} f_n(p)$

Remark: Consider $p \in bR$ s.t. $f(p) = +\infty$

$\Rightarrow p \notin D^*$

Message: If we use D^* for the contrution in Royden's compactification, the n some points of bR or Δ will be missing.