$M : \mathcal{F}(k) \to Ab \rightsquigarrow \{M_n\}$ sequence of Nisnevich sheaf with transfers
$E \mapsto M(E)$ and homotopy invariant, $(M_{n+1})_{-1} = M_n$

Now given a series of $M_n \in Sh(k)$ (perfect) which are homotopy invariant and $(M_{n+1})_{-1} = M_n$, we want to define a cycle module. Ref: Déglise, Module homotopiques avec tranfers et motif génériques.
Since k is perfect, every f.g. field E/k is a direct limit A_i of smooth k-algebras $\subseteq E$ s.t. $K(A_i) = E$. (For existence, see Déglise Lemme 2.1.32. Consider all smooth finitely generated k-algebra $A \subseteq E$. If $A, B \subseteq E$ are smooth and $K(A) = E$, then $A, B \subseteq k(A,B)$ is smooth at ξ_A so $A, B \subseteq k(A,B)_f$ smooth.) So we can define $M(E) = \varinjlim_i M(\operatorname{Spec} A_i) := \varinjlim_i \bigoplus_n M_n(\operatorname{Spec} A_i)$.

D1: $\forall\ E/F$, $M(F) \to M(E)$. Suppose E/F is an extension in $\mathcal{F}(k)$, $E = \varinjlim A_i$, $F = \varinjlim B_j$ where A_i, B_j are smooth over k. For every B_j, the composite $B_j \to F \to E = \varinjlim A_i$ factor through some A_{i_j} since B_j is finitely generated. So we define $M(F) \to M(E)$ by the diagram

$$\begin{tikzcd} M(\operatorname{Spec} B_j) \arrow[r] \arrow[d] & M(\operatorname{Spec} A_{i_j}) \arrow[d] \\ M(F) \arrow[r] & M(E) \end{tikzcd}$$

D2. E/F finite, $M(E) \to M(F)$. Suppose E/F is a finite extension and $F = \varinjlim A_i$. Take any A_i and denote by $\widetilde{A_i}$ its normalization in E. The $\widetilde{A_i}$ is a finite A_i-module and generically smooth over k. Suppose $U \subseteq \operatorname{Spec} \widetilde{A_i}$ is smooth$/k$ and $f : \operatorname{Spec}\widetilde{A_i} \to \operatorname{Spec} A_i$ is the natural map. The $f|_{f(U^c)^c}$ is finite and $f^{-1}(f(U^c)^c) \subseteq U$ is smooth$/k$.
So, we can find a finite dominant map $f : X \to Y$ in Sm/k s.t. $K(X) = E$, $K(Y) = F$ and $X \times_Y \operatorname{Spec} K(Y) = \operatorname{Spec} K(X)$.
Then we define $M(E) \to M(F)$ by the composite

$$M(E) = M(K(X)) = M(\underbrace{X \times_Y \operatorname{Spec}(K(Y))}_{\downarrow \text{ finite surj.}\ \operatorname{Spec} K(Y)}) \xrightarrow{M(f^T)} M(K(Y)) = M(F),$$

here f^T is the tranpose of the finite surj. map $X \times_Y \operatorname{Spec}(K(Y)) \to \operatorname{Spec}(K(Y))$.

D3. $K^M_k(E) \times M_n(E) \to M_{n+k}(E)$. Suppose $E = \varinjlim A_i$. For every A smooth$/k$, we define a pairing

$$\begin{aligned} \underbrace{\mathbb{Z}(\mathbb{G}_m^{\wedge 1})(\operatorname{Spec} A)}_{} \times M_n(\operatorname{Spec} A) &\to M_{n+1}(\operatorname{Spec} A) \quad (\subseteq M_{n+1}(\operatorname{Spec} A \times \mathbb{G}_m)) \\ (a, s) &\mapsto M_{n+1}(\operatorname{Spec} A \xrightarrow{(id_A, a)} \operatorname{Spec} A \times \mathbb{G}_m)(s) =: a \cdot s \end{aligned}$$

Suppose a is in the image of $\mathbb{Z}(\mathbb{G}_m^{\wedge 1})(\operatorname{Spec} A \times \mathbb{A}^1) \xrightarrow{\partial_0 - \partial_1} \mathbb{Z}(\mathbb{G}_m^{\wedge 1})(\operatorname{Spec} A)$.
Then since $\underbrace{M_{n+1}(i_0) = M_{n+1}(i_1)}$, $i_0, i_1 : \operatorname{Spec} A \to \operatorname{Spec} A \times \mathbb{A}^1$, we have $a \cdot s = 0$.

So we obtain a pairing $\underbrace{\operatorname{coker}(\mathbb{Z}(\mathbb{G}_m^{\wedge 1})(\operatorname{Spec} A \times \mathbb{A}^1) \xrightarrow{\partial_0 - \partial_1} \mathbb{Z}(\mathbb{G}_m^{\wedge 1})(\operatorname{Spec} A))} \times M_n(\operatorname{Spec} A) \to M_{n+1}(\operatorname{Spec} A)$.
Since $\mathbb{Z}(\mathbb{G}_m^{\wedge n}) \otimes \mathbb{Z}(\mathbb{G}_m^{\wedge m}) = \mathbb{Z}(\mathbb{G}_m^{\wedge n+m})$, we have a pairing

$$\underbrace{H_0(C_* \mathbb{Z}(\mathbb{G}_m^{\wedge k})(\operatorname{Spec} A))} \times M_n(\operatorname{Spec} A) \to M_{n+k}(\operatorname{Spec} A).$$

Taking limit, we obtain

$$\begin{aligned} &K^M_k(E) \times M_n(E) \to M_{n+k}(E) \\ &\ \ \| \\ &H_0(C_* \mathbb{Z}(\mathbb{G}_m^{\wedge k})(E)) \end{aligned}$$

by Thm 4.7.

To establish D4, we need an important result called homotopy purity. Suppose that $F \in Sh(k)$ is homotopy invariant and $Y \subseteq X$ is closed embedding in Sm/k. We want to understant the supported cohomology $H^*_Y(X, F)$. Recall that in prop 2.24, we have shown that for any $\varphi : Y' \xrightarrow{\text{ét}} X'$, Z closed in X' and $\varphi^{-1}(Z) = Z$, we have $H^*_Z(Y', F) = H^*_Z(X', F)$.
Since F is homotopy invariant, we have

$$H^n_Y(X, F) = \operatorname{Hom}_{D^-(k)}(\mathbb{Z}(X)/\mathbb{Z}(X \setminus Y), F[n]) = \operatorname{Hom}_{DM^{eff,-}(k)}(\mathbb{Z}(X)/\mathbb{Z}(X \setminus Y), F[n])$$

by $\mathbb{A}^1$-locality.
So it reduces to identify $\mathbb{Z}(X)/\mathbb{Z}(X \setminus Y)$ in $DM^{eff,-}(k)$.

$Y \to X \rightsquigarrow Y \to N_{Y/X}$, $Y \subseteq X$ closed.
Thm 8.20 (homotopy purity) We have $\mathbb{Z}(X)/\mathbb{Z}(X \setminus Y) = \mathbb{Z}(N_{Y/X})/\mathbb{Z}(N^{\times}_{Y/X})$ in $DM^{eff,-}(S)$.
Proof: Recall that the deformation space $D(X, Y) = Bl_{Y \times 0}(X \times \mathbb{A}^1) \setminus Bl_{Y \times 0}(X \times 0)$ admits a projection $p : D(X, Y) \to X \times \mathbb{A}^1$ and a closed embedding $i : Y \times \mathbb{A}^1 \to D(X, Y)$.
The $i|_{p^{-1}(1)}$ is the inclusion $Y \subseteq X$ so obtain a map

$$g_{X,Y} : \underbrace{\mathbb{Z}(X)/\mathbb{Z}(X \setminus Y)} \to \mathbb{Z}(D(X,Y))/\mathbb{Z}((Y \times \mathbb{A}^1)^c).$$

The $i|_{p^{-1}(0)}$ is the zero section $Y \to N_{Y/X}$ so obtain a map

$$d_{X,Y} : \underbrace{\mathbb{Z}(N_{Y/X})/\mathbb{Z}(N^{\times}_{Y/X})} \to \mathbb{Z}(D(X,Y))/\mathbb{Z}((Y \times \mathbb{A}^1)^c)$$

So it suffices to show that $g_{X,Y}$ and $d_{X,Y}$ are $\mathbb{A}^1$-weak equivalences.

Step1. Let us first consider the case for the embedding $Y \xrightarrow{0} \mathbb{A}^n \times Y$. The $Bl_{Y \times 0}(Y \times \mathbb{A}^n) \subseteq \mathbb{A}^n_Y \times_Y \mathbb{P}^{n-1}_Y$ has two properties:
1) The fiber of $\underbrace{(0 : \cdots : 0 : 1)}_{} \in \mathbb{P}^{n-1}$ is the projection $\mathbb{A}^1 \times Y \to Y$.
$\underbrace{(0, \cdots, 0, x)} \in \mathbb{A}^n$
2). The composite $E \subseteq Bl_{Y \times 0}(Y \times \mathbb{A}^1) \subseteq \mathbb{A}^n_Y \times_Y \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ is an isomorphism, where E is the exceptional divisor.
By étale excision, we have $\underbrace{\mathbb{Z}(D(X,Y))/\mathbb{Z}((Y \times \mathbb{A}^1)^c)} = \mathbb{Z}(\underbrace{Bl_{Y \times 0}(X \times \mathbb{A}^1)})/\mathbb{Z}((Y \times \mathbb{A}^1)^c)$ in $D^-(S)$.
So we consider the map $g_{Y \times \mathbb{A}^n, Y} : \underbrace{\mathbb{Z}(Y \times \mathbb{A}^n)/\mathbb{Z}(Y^c)} \to \mathbb{Z}(Bl_{Y \times 0}(Y \times \mathbb{A}^n \times \mathbb{A}^1))/\mathbb{Z}((Y \times \mathbb{A}^1)^c)$
We have a cartesian square:

$$\begin{tikzcd} (Y \times \mathbb{A}^1)^c \arrow[r] \arrow[d] & Y \times \mathbb{P}^n \setminus (0 : \cdots : 0 : 1) \arrow[d] \\ Bl_{Y \times 0}(Y \times \mathbb{A}^n \times \mathbb{A}^1) \arrow[r] & \underbrace{Y \times \mathbb{P}^n} \end{tikzcd}$$

by 1) and the horizonal maps are vector bundles. So

$$\mathbb{Z}(Bl_{Y \times 0}(Y \times \mathbb{A}^n \times \mathbb{A}^1))/\mathbb{Z}((Y \times \mathbb{A}^1)^c) = \mathbb{Z}(Y \times \mathbb{P}^n)/\mathbb{Z}(Y^c)$$

in $DM^{eff,-}$. The latter is isomorphic to $\mathbb{Z}(Y \times \mathbb{A}^n)/\mathbb{Z}(Y^c)$ by étale excision.
So $g_{Y \times \mathbb{A}^n, Y}$ is an $\mathbb{A}^1$-weak equivalence.
On the other hand, we have cartesian squares:

$$\begin{tikzcd} N^{\times}_{Y/Y \times \mathbb{A}^n} \arrow[r] \arrow[d] & Y \times \mathbb{P}^n \setminus (0 : \cdots : 0 : 1) \arrow[r, "\cong"] \arrow[d] & (Y \times \mathbb{A}^1)^c \arrow[r, "v.b"] \arrow[d] & Y \times \mathbb{P}^n \setminus (0 : \cdots : 0 : 1) \arrow[d] \\ \underbrace{N_{Y/Y \times \mathbb{A}^n}} \arrow[r] & \underbrace{Y \times \mathbb{P}^n} \arrow[r, "\cong"] & Bl_{Y \times 0}(Y \times \mathbb{A}^n \times \mathbb{A}^1) \arrow[r, "v.b"] & Y \times \mathbb{P}^n \end{tikzcd}$$

$N_{Y/Y \times \mathbb{A}^n} = Y \times \mathbb{A}^n$

where the right horizontal arrows are $\mathbb{A}^1$-weak equivalences by 2). So we have $d_{Y \times \mathbb{A}^n, Y}$ is an $\mathbb{A}^1$-weak equivalence by étale excision of the left square.

Step2: Now suppose $\varphi : U \to X$ is étale, $Y \subseteq X$ closed and $\varphi^{-1}(Y) = Y$. Then $\pi : Bl_{Y \times 0}(U \times \mathbb{A}^1) \to Bl_{Y \times 0}(X \times \mathbb{A}^1)$ and $\pi' : N_{Y/U} \to N_{Y/X}$ are étale and $\pi^{-1}(Y \times \mathbb{A}^1) = Y \times \mathbb{A}^1$ and $\pi'^{-1}(Y) = Y$. So the statement for (U, Z) and (X, Z) are equivalent by étale excision.

Step3. By [SGA1, II, Prop 4.9], there is a finite Zariski covering $X = \bigcup_i U_i$ s.t. for any i, the embedding $Y \cap U_i \to U_i$ admits a cartesian square

$$\begin{tikzcd} Y \cap U_i \arrow[r] \arrow[d] & U_i \arrow[d, "q"] \\ \mathbb{A}^{d_Y} \arrow[r, "0"] & \mathbb{A}^{d_X} \end{tikzcd}$$

where veritical arrows are étale. We want to show the statement for $(U_i, Y \cap U_i)$.
Consider the fiber product $U_i \times_{q, \mathbb{A}^{d_X}, q \times 0} ((Y \cap U_i) \times \mathbb{A}^{d_X - d_Y})$, whose fiber over $\mathbb{A}^{d_Y} \overset{0}{\subseteq} \mathbb{A}^{d_X}$ is just $(Y \cap U_i) \times_{\mathbb{A}^{d_Y}} (Y \cap U_i)$. Since the morphism $Y \cap U_i \to \mathbb{A}^{d_Y}$ is étale, the diagonal $Y \cap U_i \to (Y \cap U_i) \times_{\mathbb{A}^{d_Y}} (Y \cap U_i)$ (open and closed immersion) induces an decomposition $(Y \cap U_i) \amalg R = (Y \cap U_i) \times_{\mathbb{A}^{d_Y}} (Y \cap U_i)$.
Set $V = (U_i \times_{\mathbb{A}^{d_X}} (Y \cap U_i \times \mathbb{A}^{d_X - d_Y})) \setminus R$. Then we have two étale maps

$$p_1 : V \to U_i, \quad p_2 : V \to (Y \cap U_i) \times \mathbb{A}^{d_X - d_Y}$$

s.t. $p_1^{-1}(Y \cap U_i) = Y \cap U_i$ and $p_2^{-1}((Y \cap U_i) \times 0) = Y \cap U_i$. So we have
$\underbrace{\mathbb{Z}(U_i)/\mathbb{Z}((Y \cap U_i)^c)} = \mathbb{Z}(V)/\mathbb{Z}((Y \cap U_i)^c) = \underbrace{\mathbb{Z}((Y \cap U_i) \times \mathbb{A}^{d_X - d_Y})/\mathbb{Z}((Y \cap U_i)^c)}$.
Then we have proved the claim by step 1, 2.

Step 4: Use MV-sequence in $DM^{eff,-}$ to conclude. $\square$