$$\begin{tikzcd} N \arrow[rr, "p"] \arrow[dr] & & D_{K3} . \\ & \bar{N} \arrow[ur] & \end{tikzcd}$$

Thm: p is surjective on each connected component of N.

Twistor line: some projective lines in D_{K3},

can be geometrically realized in N.

Prop: Let (S, f) be a marked $K3$, assume $p(S,f)$ is contained in a generic twistor line $T_w \subset D_{K3}$, then there exists a unique lifting of T_w to a curve in $\bar{N}$ through (S, f),

i.e. $\exists !$ commutative diagram:

$$\begin{tikzcd} \bar{N} \arrow[r, "p"] & D \\ & T_w \arrow[u, "i"'] \arrow[ul, bend left, "\bar{i}"] \arrow[ul, phantom, "\circlearrowleft"] \end{tikzcd} \qquad \text{with} \quad (S, f) \in \bar{i}(T_w).$$

Lemma: If $\pi: X \to Y$ is a continuous map between two Hausdorff topological manifolds X, Y, assume π is locally homeomorphic, then for any connected topological space Z, and a continuous map $i: Z \to Y$, and $x_0 \in X$, $y_0 \in Y$, $z_0 \in Z$ with $\pi(x_0) = y_0$, $i(z_0) = y_0$, and two continuous maps $j_1, j_2: Z \to X$, with $j_1(z_0) = j_2(z_0) = x_0$, and

$i = \pi \circ j_1 = \pi \circ j_2$,

we must have $j_1 \equiv j_2$.

Pf:

$$\begin{tikzcd} \overset{x_0}{X} \arrow[rr, "\pi"] & & \overset{y_1}{Y} \\ & \underset{z_0}{Z} \arrow[ul, shift left, "j_1"] \arrow[ul, shift right, "j_2"'] \arrow[ur, "i"'] & \end{tikzcd}$$

$U = \{ z \in Z \mid j_1(z) \neq j_2(z) \}$.

Aim to show $U = \phi$.

X Hausdorff $\Rightarrow$ U is open

$X \to Y$ locally homeo. $\Rightarrow$ $Z \setminus U$ is open.

If $z \in Z$, $j_1(z) = j_2(z)$. Can take $j_1(z) \in V < X$,

$\pi : V \to \pi(V)$ is a homeomorphism.

take

$z \in j_1^{-1} V \cap j_2^{-1} V < Z$

then $j_1(z), j_2(z) \in V$. $\pi(j_1(z)) = \pi(j_2(z)) \in \pi(V)$

$\Rightarrow$ $j_1(z) = j_2(z)$.

So, both U, $Z \setminus U$ are open.

$z_0 \in Z \setminus U$ $\Rightarrow$ $Z \setminus U \neq \phi$.

Z connected $\Rightarrow$ $U = \phi$.

Pf of the Prop:

$\overline{N} \xrightarrow{\quad \mathcal{P} \quad} D$

α : Kähler class on S_1 $\quad$ $\mathcal{P}\langle H^{2,0}(S_1), H^{0,2}(S_1), \alpha\rangle = W_{\mathbb{C}}$.

$$\begin{array}{c} \mathcal{S}(\alpha) \\ \downarrow \\ T(\alpha) \end{array} \quad \text{twistor family.}$$

$\mathcal{P}: T(\alpha) \xrightarrow{\cong} T_W$

By Lemma: $T(\alpha) \ni (S, f)$.

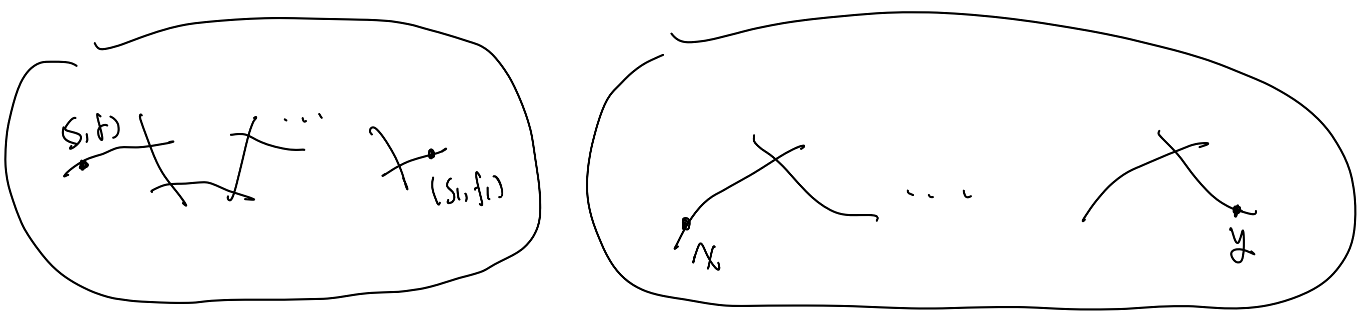
So we have the existence of a lifting of T_W.

Again by lemma, we have the uniqueness of the lifting $\square$

Pf of the surjectivity of $\mathcal{P}$:

take $x \in \mathcal{P}(N)$, take arbitrarily $y \in D_{K3}$,

$\exists$ finitely many generic twistor lines connecting x, y



N $\qquad\qquad\qquad\qquad$ D_{K3}.

$\phi(s, f) = x$.

So $y \in \phi(N)$. □

$\phi(s_1, f_1) = y$.

Next we discuss injectivity of ϕ on connected comp. of $\bar{N}$.

Lem: $\pi: X \to Y$ is a continuous map between two Hausdorff topological manifolds, assume π is locally homeomorphic,

open ball

if for any connected open subset $B \subset Y$, and any connected component C of $\pi^{-1}(\bar{B})$, one has $\pi(C) = \bar{B}$, then π is a covering map.

[$\mathbb{R}^2 - \{0\} \hookrightarrow \mathbb{R}^2$ improper.]

Lem: $D = \mathbb{P}\{x \in \Lambda_{\mathbb{C}} \mid \varphi(x,x) = 0,\ \varphi(x,\bar{x}) > 0\}$

for any connected open subset $B \subset D$, any two points $x, y \in B$,

there exists finitely many generic twistor lines $T_1, T_2, \dots, T_k$,

and $x_1 = x,\ x_2, \dots, x_k,\ x_{k+1} = y$,

such that x_i, x_{i+1} lie on a connected component of $T_i \cap B$.

[call x, y are equivalent as points in B].

Pf: it suffices to show an equivalence orbit is open.

$x = a+bi$ $\qquad \langle a,b\rangle \subset \Lambda_{\mathbb{R}}$.

take $c \in \Lambda_{\mathbb{R}}$, $\langle a,b,c\rangle$ positive 3-subspace.

take $\varepsilon \in \mathbb{R}^+$ small enough, then

$\langle a,b\rangle$, $\langle a, b+\varepsilon c\rangle$ are equivalent as points in B.

denote $d = b+\varepsilon c$.

$\langle a,b\rangle$, $\langle a,d\rangle$ lie on a connected component of

$T_{\langle a,b,d\rangle} \cap B$.

for b' near b, we have $\langle a,b'\rangle \sim \langle a,d\rangle$ as points in B

$\Rightarrow \langle a,b\rangle \underset{\text{in } B}{\sim} \langle a,b'\rangle$

for a' near a, $\langle a,b'\rangle \underset{\text{in } B}{\sim} \langle a',b'\rangle$. $\square$

Lem: $B \subset D$ open ball in a chart, $x \in \partial B$, then $\exists$ a generic twistor line $T \ni x$, $T \cap B \neq \varphi$.

Thm: $p: \bar{N} \to D$ is a covering map.

Pf: take $B \subset D$ open ball,

C: connected component of $p^{-1}(\bar{B})$.

$\exists\ (s,f) \in C,\quad \phi(s,f) = x \in B,$

$\forall\ y \in B$, we know $x \underset{\text{in } B}{\sim} y$,

$X = X_1,\ X_2, \cdots, X_{K+1} = y,$

X_i, X_{i+1} lie on a conn. comp. of $T_i \cap B$.

T_1 can be uniquely lifted to a curve $\underset{j(T_1)}{\vee}$ in $\bar{N}$, such that

$(S,f) \in j(T_1)$

$T_1 \cap B \supset K$ connected, $\quad X = X_1,\ X_2 \in K$

$j(x_1), j(x_2) \in j(K) \subset C \quad \Rightarrow \pi(C) \ni X_2.$

$\left.\begin{array}{l} j(k) \text{ conn.} \\ j(k) \ni (s,f) \\ j(k) \subset \pi^{-1}(B) \end{array}\right. \Rightarrow \pi(C) \ni X_1, X_2, \cdots, X_{K+1} = y,$

So $\pi(C) \supset B$.

For $y \in \partial B$, by the last lemma above, $\exists$ a generic twistor line connecting y with a point in B.

$\Rightarrow y \in \pi(C)$.

So $\pi: C \to \bar{B}$ is surjective

$\Rightarrow \phi: \bar{N} \to D$ is a covering map.

Fact: D_{K3} is simply connected.

Cor: Each connected component of $\bar{N}$ is mapped biholomorphically to D_{K3} via $\wp$.

Rmk: By surjectivity of $\wp: N \to D$, we know that for any decomposition $(\Lambda_{K3})_{\mathbb{C}} = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$, with

$\varphi(x,\bar{x}) > 0$, $\varphi(x,x) = 0$ for $x \in H^{2,0} - \{0\}$.

and $H^{0,2} = \overline{H^{2,0}}$,

there exists a K3 surface S with marking $f: H^2(S,\mathbb{Z}) \to \Lambda_{K3}$, such that $f(H^{2,0}(S)) = H^{2,0} \subset (\Lambda_{K3})_{\mathbb{C}}$.

Rmk: By injectivity of $\wp$ on a connected component of $\bar{N}$, If S_1, S_2 are two complex K3 surfaces such that $\exists$ $H^2(S_1,\mathbb{Z}) \xrightarrow{\sim} H^2(S_2,\mathbb{Z})$, mapping $H^{2,0}(S_1)$ to $H^{2,0}(S_2)$, then S_1, S_2 are biholomorphic.

too early to claim this.

- Next we discuss the monodromy group of complex K3, and conclude $\bar{N}$ (N) has two connected components

- Thm (global Torelli theorem): [Rappoport, Burns 198?].

Two complex K3 surfaces S_1, S_2 are isomorphic if and only if there exist an isomorphism $H^2(S_1,\mathbb{Z}) \xrightarrow{\sim} H^2(S_2,\mathbb{Z})$, $H^{2,0}(S_1) \to H^{2,0}(S_2)$. called Hodge isometry.

Moreover, for any Hodge isometry

$\psi: H^2(S_1, \mathbb{Z}) \longrightarrow H^2(S_2, \mathbb{Z})$ with $\psi(K_{S_1}) \cap K_{S_2} \neq \phi$,

then $\exists!$ isom. $f: S_2 \to S_1$, s.t. $\psi = f^*$.