Similarly, from any Lie algebra $\mathfrak{g}$ (over k of char 0), you can get the "completion" $\hat{\mathfrak{g}}_*$ which is the group defined using $\mathrm{Rep}_*(\mathfrak{g})$.

completion type

Rk: I guess that we recover the Lie group if $\mathfrak{g}$ is good enough

for example, take $\mathfrak{g}$ f.d. simple Lie alg. over $\mathbb{C}$.

then $\hat{\mathfrak{g}}_{red} = \hat{\mathfrak{g}}_{alg} = G$ Lie group ?

More examples of Hopf algebra (non-trivial one, i.e. non-commutative and non-cocommutative)

Let $\mathfrak{g}$ be a Lie algebra, and $U(\mathfrak{g})$ its universal envelopping

(i.e. "largest" alg $A \overset{i}{\hookleftarrow} \mathfrak{g}$ s.t. $\langle i(\mathfrak{g}) \rangle_{alg} = A$, and

$i([a,b]) = [i(a), i(b)] := i(a)i(b) - i(b)i(a)$)

There is a Hopf alg. structure on $U(\mathfrak{g})$, where Δ is given as follows:

$$\Delta(x) = x \otimes 1 + 1 \otimes x \qquad \text{with } x \in i(\mathfrak{g}).$$

From now, we will identify $i(\mathfrak{g})$ with $\mathfrak{g}$.

Moreover, $\boxed{\mathrm{Rep}(U(\mathfrak{g}))}$ is a tensor cat. $\boxed{\simeq \mathrm{Rep}(\mathfrak{g})}$.

(Rk: The universal enveloping "may" be defined as recovering $\mathrm{Rep}(\mathfrak{g})$ via "some" reconstruction theorem ...)

Def: An element x of a bialgebra H is called primitive if

$$\Delta(x) = x \otimes 1 + 1 \otimes x.$$

Let $\mathrm{Prim}(H)$ be the space of primitive elements.

Exercices: Show that: • $(\mathrm{Prim}(H), [\cdot,\cdot])$ is a Lie algebra.

• $x \in \mathrm{Prim}(H) \Rightarrow \begin{cases} \varepsilon(x) = 0 \\ S(x) = -x \quad (\text{if } \exists \text{ antipode } S). \end{cases}$

Prop: $\mathrm{Prim}(U(\mathfrak{g})) = \mathfrak{g}$, where $\mathfrak{g}$ Lie alg. over $\mathbb{k}$ of char 0.

Rk: Above Prop is false in general in char > 0.

Def: An element $g \in H$ is called group-like if
$$\Delta(g) = g \otimes g$$

(the notation comes from the fact that the Hopf alg. structure on $\mathbb{C}G$, with G group, is given by $\Delta(g) = g \otimes g$, $g \in G$)

Rk: the set of group-like elements forms a group.

Def: An element $x \in H$ is called <u>skew-primitive</u> if

$$\Delta(x) = g \otimes x + x \otimes h \quad \text{where } g, h \text{ are group-like.}$$

(A primitive element is skew-primitive with $g = h = 1$ $(\Delta(1) = 1 \otimes 1)$).

<u>Example</u> (Taft algebra) $\quad H_q = \langle g, x \mid g^n = 1, x^n = 0, g x g^{-1} = q x \rangle_{alg}$

where q is a nth root of unit $(q^n = 1)$ and $n \geq 2$.

<u>Thm:</u> $\dim(H_q) = n^2$

It has a Hopf alg structure with

$$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes g + 1 \otimes x$$

group-like $\qquad\qquad$ skew-primitive

It is not (co)commutative and with antipode s.t. $S^2(x) = q x \neq x$.

so $S^2 \neq id$.

Example: Hopf algebra structure on quantum group $U_q(\underline{sl_2})$.

Idea: We already know a Hopf algebra structure on universal enveloping $U(\underline{sl_2})$. Now, $U_q(\underline{sl_2})$ is a "quantum deformation" defined with $q \in k$, $q \neq 0, \pm 1$.

Recall that $\underline{sl_2}$ is generated by h, e, f s.t

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

This leads to the following def of $U_q(\underline{sl_2})$:

Def: Let $q \in k$, $q \neq 0, \pm 1$. The "quantum group" $U_q(\underline{sl_2})$ is generated by E, F, K (K invertible):

$$KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}$$

Heuristically: $K = q^h$, so that $\lim\limits_{q\to 1} \frac{K - K^{-1}}{q - q^{-1}} = h$

then, $\lim\limits_{q \to 1} U_q(\underline{sl_2})$ "=" $U(\underline{sl_2})$ (this statement can be made precise ..., see the book of Kassel)

That's why $U_q(\underline{sl_2})$ is called quantum deformation of $U(\underline{sl_2})$, by q.

$\longrightarrow$ you should think q as $\hbar$ (Planck constant) in quantum physics.
when $\hbar \to 1$, quantum mechanics becomes classical mechanics.

Thm: $\exists!$ Hopf algebra structure on $U_q(\underline{sl_2})$ with

$$\Delta(K) = K \otimes K \qquad \text{(group-like)}$$

$$\left.\begin{aligned} \Delta(E) &= E \otimes K + 1 \otimes E \\ \Delta(F) &= F \otimes 1 + K^{-1} \otimes F \end{aligned}\right\} \text{(skew-primitive)}$$

with $\varepsilon(K) = 1$, $\varepsilon(E) = \varepsilon(F) = 0$, $S(K) = K^{-1}$, $S(E) = -EK^{-1}$, $S(F) = -KF$

Assume that q is a ℓ^{th} root of unit with $\ell \geq 3$.

Let $u_q(\underline{sl_2})$ be the quotient of $U_q(\underline{sl_2})$ by relations:

$$E^\ell = F^\ell = K^\ell - 1 = 0.$$

It inherits Hopf alg structure.

It is called small quantum group attached to $\underline{sl_2}$

I guess that E, K generated a Taft subalg of $u_q(\underline{sl_2})$.

$\dim_k (u_q(\underline{sl_2})) = \ell^3$

Example/Remark: Above constructions from $\underline{sl_2}$ can be generalized to every finite dim. simple Lie alg. $\mathfrak{g}$, using its Cartan matrix

$\longrightarrow$ denoted $U_q(\mathfrak{g}), \ldots, u_q(\mathfrak{g}), \ldots$

Finite tensor categories and reconstruction theorem for quasi-Hopf algebras.

Rk: Above previous reconstruction theorem: you should not think that only finite tensor cat. $\mathcal{C}$ admits a (quasi)-fiber functor.
We will see that if it exists then the FPdim of simple objects are exactly the $\dim_{\mathbb{k}}$ of the corresponding irr. rep. of a (quasi)-Hopf alg.
In particular, if $\exists$ simple object $X \in \mathcal{C}$ s.t. $\mathrm{FPdim}(X) \notin \mathbb{Z}$. (which is the generic case) then $\mathcal{C}$ cannot have a (quasi)-fiber functor.
In other words, most of the (finite) tensor categories have no (quasi) fiber functor.

Now, the reconstruction theorem we've mentioned before involved a fiber functor. By relaxing to quasi-fiber functor, we can get the (new) notion of quasi-bialgebra, quasi-Hopf algebra.

One interesting application of this new notion is the following:

(in the semisimple case, this means $\mathrm{FPdim}(X) \in \mathbb{Z}$ $\forall X \in \mathcal{C}$ simple obj).

Prop: A finite tensor cat. $\mathcal{C}$ is integral iff $\mathcal{C}$ is equivalent to the representation cat. of a finite dim. quasi-Hopf alg.

Rk: In the previous reconstruction thm, all the finite tensor cat are integral. (most of the finite tensor cat. are not integral).

Rk: in the case of the Prop, a quasi-fiber functor always exist, and two ones are twist equiv, leading to a reconstruction thm w/o mention of a quasi-fiber functor. (in the integral, f.d. case).

See you later...