$K^M_n(k(X)) \xrightarrow{\partial} \bigoplus_{x\in X^{(1)}} K^M_{n-1}(k(x))$

Prop 8.10 The $A^0(-;K^M_n)=\underline{K}^M_n(-)$ is a Nisnevich sheaf with transfers and homotopy invariant.

Proof. It is clearly a Zariski sheaf. Suppose that we have a Nisnevich covering $p: U\to X$ and X is integral. If U is connected, then $k^M_n(X)=k^M_n(U)$. If we have $a\in k^M_n(U)$ s.t. $\partial|_{U\times U}=0$. Every component U_i of U gives a $a_i\in k^M_n(U_i)$. Then $\partial|_{U\times U}=0$ just says that these a_i are equal on intersections. □ $K^M_n(k(X))$

Prop 8.11 We have $H^p_{Nis}(X;k^M_n)=A^p(X;k^M_n)=H^p_{Zar}(X;k^M_n)$ for $X\in Sm/k$.

Proof. The Rost complex gives us a complex of sheaves on X:

$$0\to \underline{k}^M_n \to \bigoplus_{x\in X^{(0)}} k^M_n(k(x)) \to \bigoplus_{x\in X^{(1)}} k^M_{n-1}(k(x)) \to \cdots$$

The Thm 8.4 shows that this is an exact complex of Zariski sheaves.
$A^p(X;k^M_n)=0$ if X is smooth local $p>0$
Since $k^M_n(k(x))$ is flasque, we have $H^p_{Zar}(X;k^M_n)=A^p(X;k^M_n)$. The other equality comes from Prop 7.8 and Prop 8.10. □

The methods we adopted, for example, Milnor K-sheaf and its Rost complex, can be generalized to the notion of cycle modules (See [Rost, Chow group with coefficients]), which turns out to be equivalent to the heart of DM. We sketch this theory in the sequel.

Fix a base field k and consider all fields finitely generated over k, denoted by $F(k)$.

Def 8.12 A cycle premodule M consists of a function $M: F(k)\to Ab$ with a $\mathbb{Z}$-grading $M=\bigoplus_n M_n$ with data D1–D4 and rules R1a–R3e:

D1: For each $\varphi: F\to E$, there is $\varphi_*: M(F)\to M(E)$ of degree 0.
D2: For each finite $\varphi: F\to E$ there is $N_{E/F}: M(E)\to M(F)$ of degree 0.
D3: For each F the group $M(F)$ is equipped with a left $K^M_*(F)$-module structure denoted by $x\cdot\rho$ for $x\in K^M_*(F)$ and $\rho\in M(F)$ with $K^M_n(F)\cdot M_m(F)\subseteq M_{n+m}(F)$.
D4: For a discrete valuation v on F there is $\partial_v: M(F)\to M(\kappa(v))$ of degree -1.

For a uniformizer π of v on F, we put

$$s^\pi_v: M(F)\to M(\kappa(v)) \quad \text{degree } 0$$
$$\rho\mapsto \partial_v([-\pi]\cdot\rho)$$

R1a: For $F\xrightarrow{\varphi}E\xrightarrow{\psi}L$ one has $(\psi\circ\varphi)_*=\psi_*\circ\varphi_*$.
R1b: For finite extensions $F\to E\to L$, we have $N_{L/F}=N_{E/F}\circ N_{L/E}$.
R1c: For E/F finite and L/F, we have a commutative diagram

$$\begin{tikzcd} M(E) \arrow[r, "((\varphi_p)_*)_p"] \arrow[d, "N_{E/F}"'] & \bigoplus_{p\in \operatorname{Spec}(E\otimes_F L)} M(R(p)) \arrow[d, "\sum N_{R(p)/L}"] \\ M(F) \arrow[r] & M(L) \end{tikzcd}$$

R2: For $\varphi: F\to E$, $x\in K^M_*(F)$, $y\in K^M_*(E)$, $\rho\in M(F)$, $\mu\in M(E)$, one has
R2a: $\varphi_*(x\cdot\rho)=\varphi_*(x)\cdot\varphi_*(\rho)$.
R2b: If E/F is finite, $N_{E/F}(\varphi_*(x)\cdot\mu)=x\cdot N_{E/F}(\mu)$.
R2c: — — — $N_{E/F}(y\cdot\varphi_*(\rho))=N_{E/F}(y)\cdot\rho$.
R3a: Let $\varphi: E\to F$ and v be a DV on F whose ramification index w.r.t. E is e. Then we have a diagram

$$\begin{tikzcd} M(E) \arrow[r, "\partial_{v|_E}"] \arrow[d, "\varphi_*"'] & M(\kappa(v|_E)) \arrow[d, "e\cdot\bar\varphi_*"] \\ M(F) \arrow[r, "\partial_v"] & M(\kappa(v)) \end{tikzcd}$$

R3b: Suppose E/F is finite and $v\in DV(F)$. We have a diagram:

$$\begin{tikzcd} M(E) \arrow[r, "(\partial_w)"] \arrow[d, "N_{E/F}"'] & \bigoplus_{w/v} M(\kappa(w)) \arrow[d, "\sum N_{\kappa(w)/\kappa(v)}"] \\ M(F) \arrow[r, "\partial_v"] & M(\kappa(v)) \end{tikzcd}$$

R3c: Let $\varphi: E\to F$ and $v\in DV(F)$ s.t. $v|_E=0$. Then

$$\partial_v\circ\varphi_*=0.$$

R3d: φ, v as above$^{(R3c)}$, we have a diagram

$$\begin{tikzcd} M(E) \arrow[r, "\varphi_*"] \arrow[dr, "\bar\varphi_*"'] & M(F) \arrow[d, "s^\pi_v"] \\ & M(\kappa(v)) \end{tikzcd}$$

R3e. For $v\in DV(F)$, $u\in O^*_v$ and $\rho\in M(F)$, one has

$$\partial_v([u]\cdot\rho)=-[\bar u]\cdot\partial_v(\rho).$$

Def 8.13 A pairing $M\times M'\to M''$ of cycle premodules is given by bilinear maps for each F in $F(k)$

$$M(F)\times M'(F)\to M''(F)$$
$$(\rho,\ \mu)\mapsto \rho\cdot\mu$$

satisfying P1–P3 stated below.

P1: For $x\in K^M_*(F)$, $\rho\in M(F)$, $\mu\in M'(F)$ one has
P1a: $(x\cdot\rho)\cdot\mu = x\cdot(\rho\cdot\mu)$
P1b: $(-1)^{\deg x \deg\rho}\rho\cdot(x\cdot\mu)=x\cdot(\rho\cdot\mu)$
P2: For $\varphi: F\to E$, $\eta\in M(F)$, $v\in M(E)$, $\rho\in M'(F)$, $\mu\in M'(E)$, one has:
P2a: $\varphi_*(\eta\cdot\rho)=\varphi_*(\eta)\cdot\varphi_*(\rho)$
P2b: E/F finite, $N_{E/F}(\varphi_*(\eta)\cdot\mu)=\eta\cdot N_{E/F}(\mu)$
P2c: - - - - -, $N_{E/F}(v\cdot\varphi_*(\rho))=N_{E/F}(v)\cdot\rho$.
P3: For $v\in DV(F)$, $\eta\in M_n(F)$, $\rho\in M(F)$ and π is a uniformizer of v, one has

$$\partial_v(\eta\cdot\rho)=\partial_v(\eta)\cdot s^\pi_v(\rho)+(-1)^n s^\pi_v(\eta)\cdot\partial_v(\rho)+\partial_v(\eta)\partial_v(\rho)[-1].$$

A ring structure on M is a pairing $M\times M\to M$ which is associative and graded commutative.

$\tilde Z \ni y_i$, $\tilde Z \xrightarrow{p} Z=\bar X$, $y_i \mapsto y$

Then for any scheme X/k, $x\in X$, $y\in \bar x^{(1)}$, $Z=\bar X$, we define

$$\partial^x_y: M(k(x))\to M(k(y))$$
$$\rho\mapsto \sum_{p(y_i)=y} N_{k(y_i)/k(y)}(\partial_i(\rho))$$

Def 8.14 A cycle module M is a cycle premodule M satisfying the following conditions (FD) and (C):
(FD): Let X be a normal scheme and $\rho\in M(K(X))$. Then $\partial^{\xi_X}_x(\rho)=0$ for all but finite many $x\in X^{(1)}$.
(C): Let X be integral and local of dimension 2. Then

$$\sum_{x\in X^{(1)}} \partial^x_{m_X}\circ\partial^{\xi_X}_x=0.$$

Prop. 8.15 Suppose that M is a cycle module. Then we have the followings:
(H) The sequence $0\to M(F)\xrightarrow{i} M(F(t))\underset{\tau_x}{\overset{\partial_x}{\rightleftarrows}}\bigoplus_{x\in A^1_F} M(k(x))\to 0$ is split exact.
(RC) Let X be a proper curve $/F$. Then

$$\sum_{x\in X^{(1)}} N_{k(x)/F}\circ\partial^{\xi_X}_x=0.$$

Proof: (H) We define $\tau_x: M(k(x))\to M(F(t))$

$$\mu\mapsto N_{k(x)(t)/F(t)}([t-t(x)]\cdot(i_x)_*(\mu))$$

$k(x)\xrightarrow{i_x}k(x)(t)$; $t(x)$: generator of $k(x)/F$

We have a section $s^0: M(F(t))\to M(F)$. Then one checks directly.
(RC) Find a finite map $X\to\mathbb{P}^1$, proceeding as in Thm 3.12 and use (H). □

We define $M(X)$ by the exact sequence

$$0\to M(X)\to M(K(X))\xrightarrow{\partial^{\xi_X}}\bigoplus_{x\in X^{(1)}} M(k(x))$$

for every integral scheme X/k.
Also, we have the Rost complex

$$C^n(X;M)=\bigoplus_{x\in X^{(n)}} M(k(x))$$

$$\bigoplus_{x\in X^{(0)}} M(k(x))\xrightarrow{\partial^x_y}\bigoplus_{x\in X^{(1)}} M(k(x))\to\cdots$$

hence we define $A^p(X;M)=H^p(C^*(X;M))$.

Prop 8.16 Suppose M is a cycle module.
1) $A^p(-;M)$ is a presheaf with transfers and $M(-)$ is a Nisnevich sheaf.
2) $A^p(-;M)$ is homotopy invariant.
3) $A^p(X;M)=H^p_{Nis}(X;M)=H^p_{Zar}(X;M)$.

Proof: For every flat morphism $f: X\to Y$, we can define a pull-back $f^*: C^*(Y;M)\to C^*(X;M)$ as in Def 3.16. It induces a pull-back $f^*: A^p(Y;M)\to A^p(X;M)$. For every proper morphism $f: X\to Y$, we can define its push-forward $f_*: A^p(X;M)\to A^{p+d_Y-d_X}(Y;M)$ as in Def 3.15. Then the 2) comes from the same proof as in Prop 8.1. Then the deformation to normal bundle gives us the Gysin pullback along closed immersions. Then 1) comes from the same proof as in Prop 8.9, 8.10. Finally 3) comes from the same proof as in Thm 8.4 and Prop 8.11. □

Prop 8.17 The $M(F)=\bigoplus_{n\in\mathbb{N}} K^M_n(F)$ is a cycle module. Galois cohomology, Quillen's K-theory [Rost]
Proof: Follows from all results in Sect 3. □

Def 8.18 Suppose $F\in PSh(k)$ is homotopy invariant. Define $F_{-1}(X)$ for any $X\in Sm/k$ by the exact sequence

$$\underbrace{F(X\times\mathbb{A}^1)}_{=F(X)}\to F(X\times\mathbb{G}_m)\to F_{-1}(X)\to 0.$$

Since $F(X)=F(X\times\mathbb{A}^1)$ and $F(X)\to F(X\times\mathbb{G}_m)$ has a section i_1^*, we have $F(X\times\mathbb{G}_m)=F(X)\oplus F_{-1}(X)$ and $F_{-1}(X)=\ker(F(X\times\mathbb{G}_m)\xrightarrow{i_1^*}F(X))$.

Prop 8.19 Suppose M is a cycle module. We have $(M_{n+1})_{-1}=M_n$.
Proof. We have an exact sequence

$$A^0(X\times\mathbb{A}^1;M_{n+1})\to A^0(X\times\mathbb{G}_m;M_{n+1})\xrightarrow{\partial(X\times 0\subset X\times\mathbb{A}^1)}A^0(X;M_n)\xrightarrow{i_0*}A^1(X\times\mathbb{A}^1;M_{n+1})$$

so it induces a map $(M_{n+1})_{-1}\to M_n$. When $X=\operatorname{Spec}F$ of a field F, we have $A^1(\mathbb{A}^1_F;M_{n+1})=A^1(F;M_{n+1})=H^1(F;M_{n+1})=0$. So ∂ is surjective. Hence $(M_{n+1})_{-1}(F)=M_n(F)$. Then use Thm 6.7 to conclude. □

$\underline{M: F(k)}\to\underline{Ab}\rightsquigarrow M_n$: homotopy invariant Nisnevich sheaf with transfers

$\underline{(M_{n+1})_{-1}=M_n}$