Limit (and colimit) in category theory:

What we called inverse limit (denoted $\varprojlim$) or projective limit is just called *limit* in general category theory, whereas (the dual) *colimit* corresponds to what we called direct limit (denoted $\varinjlim$) or inductive limit.

Let me define the notion of limit (I used Wikipedia) -

Let J be a category (think it as an index category), $G: J \to C$ a functor (think it as indexing a collection of objects and morphisms in the category C), it is called a *diagram* of shape J.

A *cone* to G is an object $N \in C$ together with a family of morphisms $\psi_X : N \to G(X)$ indexed by the objects $X \in J$, s.t.

$\forall$ morphism $f : X \to Y$ (in J), $\boxed{G(f) \circ \psi_X = \psi_Y}$.

A limit of the diagram $G : J \to C$ is a cone (L, ϕ) to G satisfying a universal property as follows:

for every other cone (N, ψ) to G, $\exists! \, u : N \to L$ s.t. $\phi_X \circ u = \psi_X \quad \forall X \in J$

$$\begin{tikzcd}
 & N \arrow[d, dashed, "\exists! u"] \arrow[ddl, bend right, "\psi_X"'] \arrow[ddr, bend left, "\psi_Y"] & \\
 & L \arrow[dl, "\phi_X"'] \arrow[dr, "\phi_Y"] & \\
G(X) \arrow[rr, "G(f)"'] & & G(Y)
\end{tikzcd}$$

in other words: let $\varprojlim_{X \in J} G(X) := L$. Then:

$$\varprojlim_{X \in J} \hom_C(N, G(X)) \simeq \hom_C\left(N, \varprojlim_{X \in J} G(X)\right)$$

Rk: it can be used directly to define L (up to some representable assumption) ← i.e. rep. functor using hom...)

Colimit: is the dual notion of limit, so defined by reversing all the arrows in above diagram, i.e. a co-cone (L, ϕ) satisfying the universal property making the following diagram commute

$$\begin{tikzcd}
G(X) \arrow[rr, "G(f)"] \arrow[dr, "\phi_X"] \arrow[ddr, bend right, "\psi_X"'] & & G(Y) \arrow[dl, "\phi_Y"'] \arrow[ddl, bend left, "\psi_Y"] \\
& L \arrow[d, dashed, "u"] & \\
& N &
\end{tikzcd}$$

in other words, let $\varinjlim_{X \in J} G(X) =: L$

$$\varprojlim_{X \in J} \hom_C(G(X), N) \simeq \hom_C(\varinjlim_{X \in J} G(X), N)$$

here it's limit (not colimit)! / + same Rk: ...

Intuition: When J is small category (the objects $Ob(J)$ and the morphisms $hom(J)$ are sets)

then limit can be defined as a subset of $\prod_{X \in J} \dots$ (product)

colimit ——— quotient of $\coprod_{X \in J} \dots$ (coproduct, i.e. disjoint union here)

EXO

Now, let us come back to ring category $\mathcal{C}$ with a fiber functor F:

Then: what is noted $\varprojlim \mathrm{End}\, F(X)$ in Section 1.10 of the book

should be defined as the limit of the diagram $G : J \to C$

$J =$ diagonal of $\mathcal{C} \boxtimes \mathcal{C}^\vee$

$C = \mathrm{Vec}$

$G = F \boxtimes F^\vee$

where $\boxtimes$ Deligne tensor product

$\mathcal{C}^\vee$: dual category (i.e. reversing the arrows, i.e. $\hom_{\mathcal{C}^\vee}(X,Y) = \hom_{\mathcal{C}}(Y,X)$)

Then, $G(X \boxtimes X) \simeq \mathrm{End}(F(X))$

I guess ... check it ... as an exo.

Finally, as another exo, the commuting diagram defining cone and limit, will exactly corresponds to the commuting diagrams defining a natural transformation $\phi: F \to F$, so that the limit of $G: J \to C$ is exactly $End(F)$.

⚠ I wrote this sketch quickly, without checking the details ..., so you should NOT take it for granted, but check the details as an exercise.

Now, because $End(F)$ may be infinite dim. in general, then

$$End(F) \otimes End(F) \subsetneq End(F \boxtimes F)$$

$\downarrow$ completion with respect to a topology induced by the inverse limit definition of $End(F) = \varprojlim End(F(X))$

$$End(F) \hat{\otimes} End(F) = End(F \boxtimes F)$$

The comultiplication of $\mathrm{End}(F)$ is a continuous linear map

$$\Delta : \mathrm{End}(F) \to \mathrm{End}(F) \hat{\otimes} \mathrm{End}(F)$$

Next: $\mathrm{Coend}(F) := \varinjlim \mathrm{End}(F(X))^*$

(colimit)

Exo: write the details for defining $\mathrm{Coend}(F)$

it inherits of a coalgebra structure by def.

The dual Δ^* defines a multiplication on $\mathrm{Coend}(F)$

ε^* unit

So, $\mathrm{Coend}(F)$ has a structure of bialgebra.

Rk: If $\mathcal{C}$ has left duals, then the bialgebra Coend(F) has an antipode, defined as before.

I guess: it does not work for End(F)?

that's why the reconstruction thm in co_care needs Coend(F) instead of End(F),

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This antipode is invertible if $\mathcal{C}$ is rigid (i.e. also with right duals)

Reconstruction Theorem: the assignments $(\mathcal{C}, F) \mapsto H = \mathrm{Coend}(F)$

$H \mapsto (H\text{-Comod}, \text{forget})$

are mutually inverse bijection between:

(1) ring cat. $\mathcal{C}$ (over k) with a fiber functor F (up to ...) and bialgebra H (over k), up to ...

(2) ... idem with left duals . . . idem with antipode ....

(3) .. idem .. with also right duals (i.e. rigid, i.e. tensor cat) .... idem Hopf algebra ...

Rk 1: Above thm permits a categorical proof of
"if antipode exists then it is unique" (w/o assuming finite)
using "if left dual exists, then it is unique, up to ~".

Rk 2: $\exists$ ∞-dim bialgebra H with a non-invertible antipode
(i.e. some previous prop in the finite dim case does not pass to ∞-case)
$\longleftrightarrow$ $\exists$ ring cat. (with simple 1) and left duals but w/o right duals
(i.e. not tensor). For example: H-comod.

History about reconstruction theorems for groups G and Rep(G).

- on (locally) compact topological abelian groups:
  - Pontryagin (1934): compact or discrete abelian groups.

- Van Kampen (1935), Weil (1940): general locally compact abelian groups

- compact non-abelian top. groups: Tannaka (1939) / Krein (1949)

- algebraic group: Cartier (1956) / Grothendieck's school...
Saavedra Rivano (1972)
Deligne, Milne (1982)

↳ Tannakian category theory

proalgebraic group G ⇄ cat. $\mathcal{C}$ of alg. rep. of G + forgetful functor F.

$G = \operatorname{Spec} \mathcal{O}(G)$

$\operatorname{Coend}(F)$
$=$
$\mathcal{O}(G)$: Hopf alg of regular functions on G

The use of such reconstruction thm permits to provide various completion $\hat{G}_*$ of any given (discrete) group G, according to the type $*$ of rep you choose: $f_* : \mathrm{Rep}_*(G) \to \mathrm{Vec}$, $\hat{G}_* := \mathrm{Spec}\,\mathrm{Coend}(f_*)$

$\mathrm{Rep}_*(G) = \mathcal{C}_*$

type of rep	completion name	notation
finite dim. over $\Bbbk$	proalgebraic	$\hat{G}_{alg}$ (i.e. $*$ = alg)
semisimple f.d. — char 0	proreductive	$\hat{G}_{red}$
triangular rep (i.e. f.d. with 1-dim factors)	prosolvable	$\hat{G}_{solv}$
unipotent rep. (i.e. f.d. with trivial factor)	prounipotent	$\hat{G}_{unip}$
f.d. rep factoring through finite groups	profinite	$\hat{G}_{fin}$

for more details, see book on page 98-99 Examples 5.4.4. and the references there.

$\exists$ canonical homomorphism $\xi : G \to \hat{G}_*$

moreover $\hat{G}_*$ is inverse limit of alg. groups, so there is an inverse limit topology making $\xi(G)$ dense in $\hat{G}_*$.

Rk: $\hat{G}_*$ may be really HUGE (see $G = F_2$ free group) but can also be trivial... in fact, ξ may not be injective, for example if G has no f.d. rep ($\neq$ trivial) (for ex. finitely generated infinite simple group), then $\mathcal{C}_* = \mathrm{Vec}$, and $\hat{G}_* = 1$.

See you next time...