

limit (and colimit) in category theory:

What we called inverse limit (denoted \lim_{\leftarrow}) or projective limit is just called limit in general category theory, whereas (the dual) colimit corresponds to what we called direct limit (denoted \lim_{\rightarrow}) or inductive limit.

Let me define the notion of limit (I used Wikipedia)-

Let J be a category (think it as an index category), $G: J \rightarrow C$ a functor (think it as indexing a collection of objects and morphisms in the category C), it is called a diagram of shape J .

A cone to G is an object $N \in C$ together with a family of

morphisms $\psi_x : N \rightarrow G(x)$ indexed by the objects $x \in J$, s.t.

\forall morphism $f: x \rightarrow y$ ($\text{in } J$), $G(f) \circ \psi_x = \psi_y$.

A limit of the diagram $G: J \rightarrow C$ is a cone (L, ϕ) to G satisfying a universal property as follows:

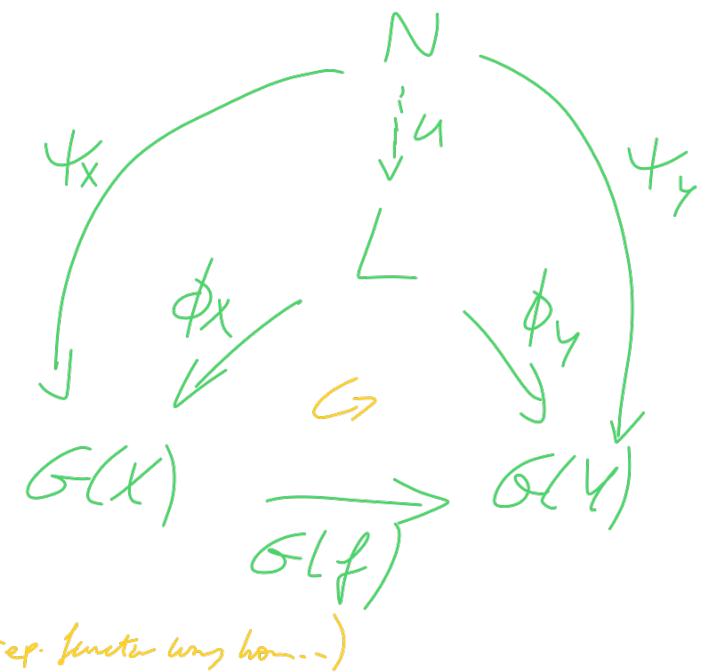
for every other cone (N, ψ) to G , $\exists ! u: N \rightarrow L$

$$\text{s.t. } \phi_x \circ u = \psi_x \quad \forall x \in J$$

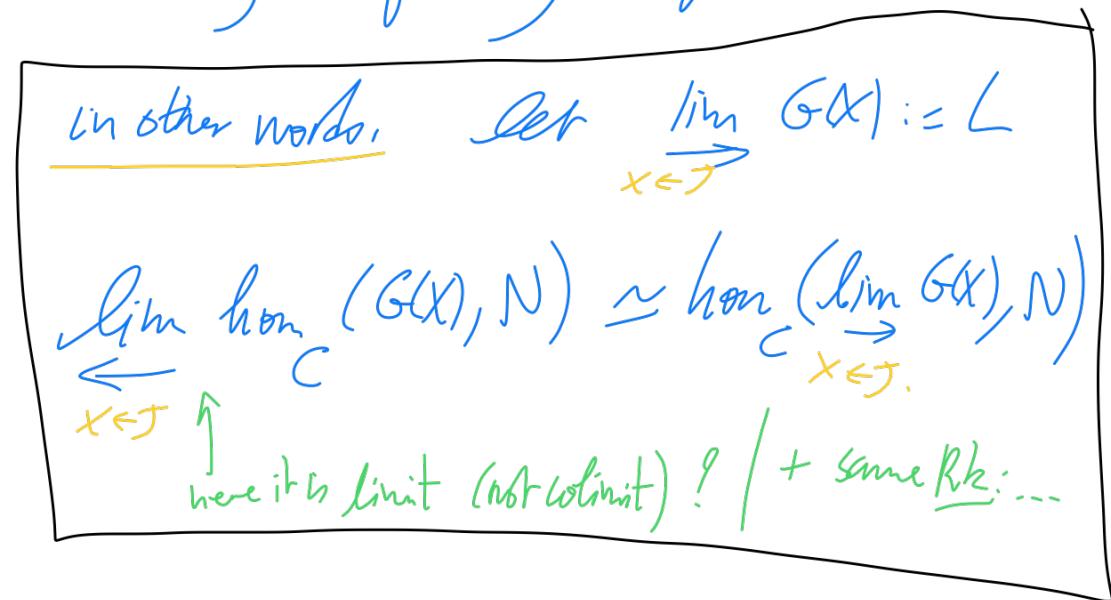
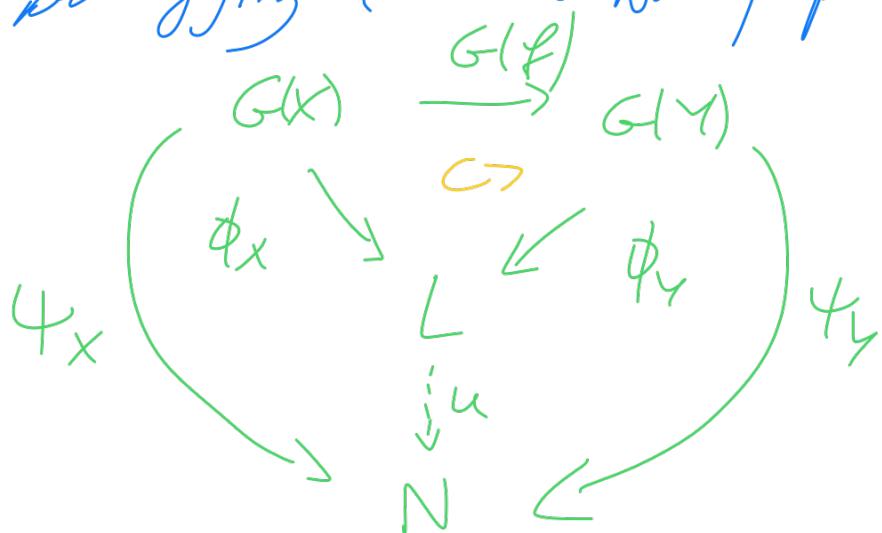
in other words: let $\lim_{\leftarrow, x \in J} G(x) := L$. Then:

$$\lim_{\leftarrow, x \in J} \text{hom}_C(N, G(x)) \simeq \text{hom}_C(N, \lim_{\leftarrow, x \in J} G(x))$$

Rn: it can be used directly to define L (upto some representable assumption \hookrightarrow)



Colimit : is the dual notion of limit , so defined by reversing all the arrows in above diagram, i.e. a co-cone (L, ϕ) satisfying the universal property making the following diagram commutes



Intuition: When J is small category (the objects $Ob(J)$ are sets)
and the morphisms $hom(J)$

then limit can be defined as a subset of $\prod_{X \in J} \dots$ (product)

colimit —————

quotient of $\coprod_{X \in J} \dots$ (coproduct, i.e. disjoint union here)

Exo

Now, let us come back to ring category \mathcal{G} with a fiber functor F :

Then: what is noted $\varprojlim \text{End}(F(X))$ in section 1.1b of the book

should be defined as the limit of the diagram $\mathcal{G}: \mathcal{T} \rightarrow \mathcal{C}$

$$\begin{array}{l} \mathcal{T} = \text{diagonal of } \mathcal{G} \boxtimes \mathcal{G}^\vee \\ \mathcal{C} = \text{Vec} \\ \mathcal{G} = F \boxtimes F^\vee \end{array} \quad \left. \begin{array}{l} \text{where } \boxtimes \text{ Deligne tensor product} \\ \mathcal{C}^\vee: \text{dual category (i.e. reversing the arrows, i.e.} \\ \text{hom}_{\mathcal{C}^\vee}(X,Y) = \text{hom}_{\mathcal{C}}(Y,X) \end{array} \right\}$$

Then, $G(X \boxtimes X) \cong \text{End}(F(X))$

I guess ... check it...
as an exo.

Finally, as another exo, the commuting diagram defining cone

and limit), will exactly corresponds to the commuting diagrams defining a natural transformation $\phi: F \rightarrow F$, so that the limit of $G: \mathcal{T} \rightarrow \mathcal{C}$ is exactly $\text{End}(F)$.

A I wrote this sketch quickly, without checking the details --,
so you should NOT take it for granted, but check the details as an exercise.

Now, because $\text{End}(F)$ may be infinite dim. in general, then

$$\text{End}(F) \otimes \text{End}(F) \not\subseteq \text{End}(F \boxtimes F)$$

↓ completion with respect to a topology induced by
the inverse limit definition

$$\text{End}(F) \widehat{\otimes} \text{End}(F) = \text{End}(F \boxtimes F)$$

$$\text{of } \text{End}(F) = \varprojlim \text{End}(F \otimes I)$$

The comultiplication of $\text{End}(F)$ is a continuous linear map

$$\Delta: \text{End}(F) \rightarrow \text{End}(F) \hat{\otimes} \text{End}(F)$$

Next: $\text{Coend}(F) := \lim_{\rightarrow} \text{End}(F(X))^*$ | Exo: write
the details for
defining $\text{Coend}(F)$

([↑] colimit)

it inherits of a coalgebra structure by def.

The dual Δ^* defines a multiplication on $\text{Coend}(F)$

ε^* unit

So, $\text{Coend}(F)$ has a structure of bialgebra.

Rk: If \mathcal{C} has left duals, then the bialgebra $\text{Coend}(F)$ has an antipode, defined as before.

I guess: it does not work for $\text{End}(F)$?

This antipode is invertible if \mathcal{C} is rigid (ie also with right duals)

(that's why the reconstruction theorem in oo-case needs $\text{Coend}(F)$ instead of $\text{End}(F)$,

Reconstruction Theorem: the assignments $(\mathcal{C}, f) \mapsto H = \text{Coend}(F)$ are mutually inverse bijection between:

$$H \mapsto (H\text{-Comod}, \text{forget})$$

(1) ring cat. \mathcal{C} (over k) with a fiber functor f (upto...) and bialgebra H (over k), up to...

(2) ... idem with left duals . . . -- - - - - - idem with antipode ...

(3) ... idem .. with also right duals (ie rigid; ie tensor cat). . . . idem Hopf algebra ...

Rk 1: Above thm permits a categorical proof of
"if antipode exists then it is unique" (w/o assuming finite)

using "if left dual exists, then it is unique, upto~".

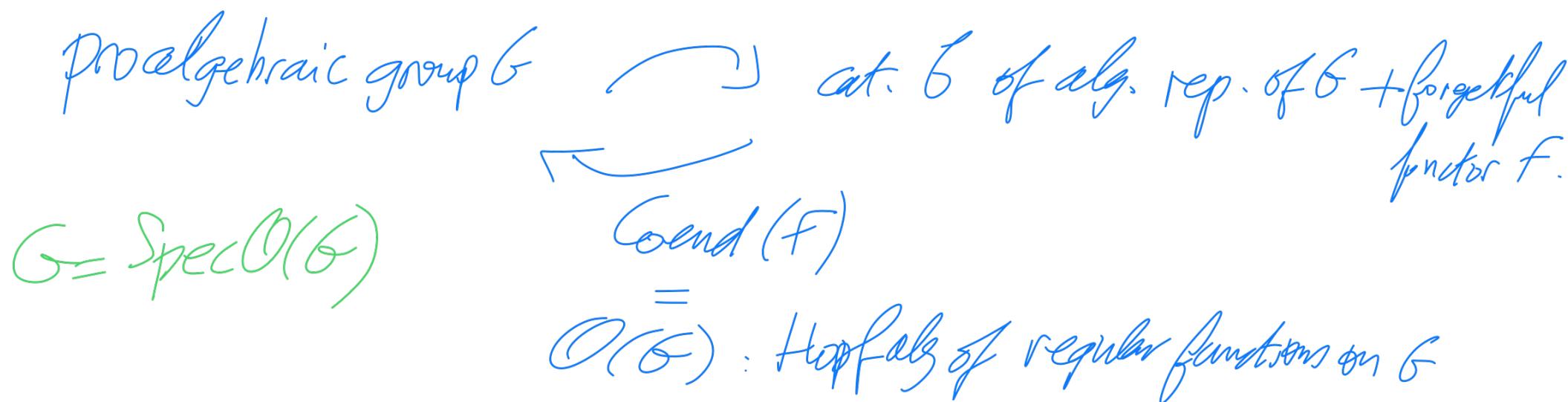
Rk 2: $\exists \infty\text{-dim bialgebra } H \text{ with a non-invertible antipode}$
(i.e. some previous prop in the finite dim case does not pass to $\infty\text{-ac}$)

$\longleftrightarrow \exists \text{ ring cat. (with simple 1) and left duals such w/o right duals}$
(i.e. not tensor). For example: $H\text{-comod}.$

History about reconstruction theorems for groups G and $\text{Rep}(G)$.

- on (locally) compact topological abelian groups:
 - Pontryagin (1934) : compact or discrete abelian grp.

- Van Kampen (1935), Weil (1940) : general locally compact abelian groups
- Compact non-abelian top. groups: Tannaka (1939) / Krein (1949)
- algebraic group : Cartier (1956) , Grothendieck's th...
 ↳ Tannakian Category theory
 Saavedra Rivano (1972)
 Deligne, Milne (1982)



The use of such reconstruction thus permits to provide various completion ^{\widehat{G}_*} of any given (discrete) group G , according to the type $*$ of rep you choose : $f_* : \text{Rep}_*(G) \rightarrow \text{Vec}$, $\widehat{G}_* := \text{Spec Colnd}(f_*)$

type of rep	completion name	notation
finite dim. over \mathbb{K}	proalgebraic	\widehat{G}_{alg} (i.e. $* = \text{aff}$)
semi-simple f.d. — char 0	proreductive	\widehat{G}_{red}
triangular rep (i.e. f.d. with 1-dim factors)	prosolvable	$\widehat{G}_{\text{solv}}$
unipotent rep. (i.e. f.d. with trivial factor)	prox unipotent	$\widehat{G}_{\text{unip}}$
f.d. rep factoring through finite groups	profinite	\widehat{G}_{fin}

for more details, see book on page 98-99 Examples 5.4.4. and the references there.

7 canonical homomorphism $\tilde{g}: G \rightarrow \hat{G}_*$

moreover \hat{G}_* is inverse limit of ab. groups, so there is
an inverse limit topology making $\tilde{g}(G)$ dense in \hat{G}_* .

Rk: \hat{G}_* may be really HUGE (see $G = \mathbb{Z}$ free group)
but can also be trivial --- in fact, \tilde{g} may not be
injective, for example if G has no f.d. rep (\neq trivial)
(for ex. finitely generated infinite simple group),
then $\hat{G}_* = \mathbb{K}^G$, and $\hat{G}_* = 1$.

See you next time....

