Twister lines

Λ: a lattice of signature $(3, b-3)$ $\quad \varphi: \Lambda \times \Lambda \to \mathbb{Z}$.

$W \subset \Lambda_{\mathbb{R}}$: positive 3-subspace.

$T_W := D \cap \mathbb{P}(W_{\mathbb{C}}) \hookrightarrow \mathbb{P}(W_{\mathbb{C}}) \cong \mathbb{P}^2$ $\quad$ (T_W is called a twistor line)

quadric curve in $\mathbb{P}^2$, isomorphic to $\mathbb{P}^1$.

$D := \mathbb{P}\{ x \in \Lambda_{\mathbb{C}} \mid \varphi(x, x) = 0, \ \varphi(x, \bar{x}) > 0 \}$.

$=$ set of oriented positive planes in $\Lambda_{\mathbb{R}}$.

$x = a + bi \longleftrightarrow \langle a, b \rangle \subset \Lambda_{\mathbb{R}}$.

T_W called generic, if $\exists\, w \in W$, s.t. $w^{\perp} \cap \Lambda = 0$

(W)

$\Longleftrightarrow \exists\, x \in T_W$, s.t. $x^{\perp} \cap \Lambda = 0$.

Def: $x, y \in D$ called equivalent iff x, y can be connected by finitely many generic twister lines.

Prop: any two points in D are equivalent.

We will use this result to show surjectivity of

$p: N \longrightarrow D_{K3}$

Let S be a complex K3 surface, I be the complex structure.

$\sigma_I \in H^0(\Omega_S^2)$ a generator,

take a Kähler metric g,

Then I, σ_I is constant tensors with respect to the Levi-Civita connection of g.

We have another two complex structures J, K on S,

such that $IJ = K$, $JK = I$, $KI = J$, $I^2 = J^2 = K^2 = -1$.

J, K are also constant tensors.

Denote ω_I the Kähler form associated with g and I,

i.e. $\omega_I(x, y) = g(Ix, y)$.

ω_I is a closed real form of Hodge type $1,1$.

$\lambda = aI + bJ + cK$, $a^2 + b^2 + c^2 = 1$, $a, b, c \in \mathbb{R}$.

is a complex structure,

(S, λ, g) is a Kähler surface,

ω_λ Kähler form asso. with λ, g,

$\omega_\lambda(x, y) = g(\lambda x, y)$.

ω_λ is also a closed real form.

$\mathbb{R}\{\omega_\lambda \mid \lambda = aI + bJ + cK\}$ is a 3-subspace of $H^2(S, \mathbb{R})$.

Lemma. $\omega_J + i\,\omega_K$ is a generator of $H^{2,0}(S)$.

with complex structure I

Pf: It suffices to check on the tangent space of one point,

take $b \in S$, consider $T_b S \cong \mathbb{H} = \mathbb{R}\{1, I, J, K\}$

$$= \mathbb{C} \oplus \mathbb{C}J.$$

$$\mathbb{C} = \mathbb{R}\{1, I\},$$

$T_b S \ni u_1 + v_1 J,\ u_2 + v_2 J,$

$u_1, v_1, u_2, v_2 \in \mathbb{C}$

then $\omega_J(u_1 + v_1 J,\ u_2 + v_2 J)$

$$= g(J u_1 + J v_1 J,\ u_2 + v_2 J)$$

$$= g(-\bar{v}_1 + \bar{u}_1 J,\ u_2 + v_2 J)$$

$$= \mathrm{Re}\left(-\bar{v}_1 \cdot \bar{u}_2 + \bar{u}_1 \cdot \bar{v}_2\right)$$

$\omega_K(u_1 + v_1 J,\ u_2 + v_2 J)$

$$= g(K u_1 + K v_1 J,\ u_2 + v_2 J)$$

$$= g(-I\bar{v}_1 + I\bar{u}_1 \cdot J,\ u_2 + v_2 J)$$

$$= \mathrm{Re}(-I\bar{v}_1 \cdot \bar{u}_2 + I\bar{u}_1 \cdot \bar{v}_2)$$

$$= \mathrm{Im}(\bar{v}_1 \cdot \bar{u}_2 - \bar{u}_1 \cdot \bar{v}_2)$$

So: $(\omega_J + I\omega_K)(u_1+v_1J,\ u_2+v_2J)$

$= \mathrm{Re}(-\bar{v}_1\bar{u}_2 + \bar{u}_1\bar{v}_2) + I\cdot \mathrm{Im}(\bar{v}_1\bar{u}_2 - \bar{u}_1\bar{v}_2)$

$= \mathrm{Re}(u_1v_2 - v_1u_2) + I\cdot \mathrm{Im}(u_1v_2 - v_1u_2)$

$= u_1v_2 - v_1u_2.$

$\Rightarrow$ $\omega_J + I\omega_K$ is the $\underbrace{\text{symplectic}}_{}$ standard symplectic form on $\mathbb{C}\oplus\mathbb{C}J$ $\square$

(S, I). take g Kähler metric.

can take $\sigma_I = \omega_J + i\omega_K \in H^0(\Omega_S^2)$ the generator,

$\langle \sigma_I, \bar{\sigma}_I, \omega_I \rangle_{\mathbb{C}} = \langle \omega_J, \omega_K, \omega_I \rangle_{\mathbb{C}}$ as subspaces of $H^2(S, \mathbb{C})$

this is positive

$\langle \omega_I, \omega_J, \omega_K \rangle_{\mathbb{R}}$ is a positive 3-subspace of $H^2(S, \mathbb{R})$.

S complex K surface, take a Kähler form α.

$\rightsquigarrow$ positive 3-subspace of $H^2(S, \mathbb{R})$ W_α.

$\rightsquigarrow$ Twistor line on $D = \mathbb{P}\{x \in H^2(S, \mathbb{C}) \mid (x,x)=0,\ (x,\bar{x})>0\}$

denoted by T_{W_α}.

Let $T(\alpha) = \{(a,b,c) \mid a^2+b^2+c^2=1,\ a,b,c \in \mathbb{R}\}$

$= S^2 \cong \mathbb{P}^1.$

$\mathcal{X}(\alpha) = S \times T(\alpha)$ with complex structure given by:

$$T_m S \oplus T_\lambda T(\alpha) \longrightarrow T_m S \oplus T_\lambda T(\alpha).$$

$$(v, w) \longmapsto (\lambda(v), I_{T(\alpha)}(w)).$$

$\mathcal{X}(\alpha)$ is complex manifold of dimension 3.

$$\begin{array}{c} \mathcal{X}(\alpha) \\ \downarrow \\ T(\alpha) \cong \mathbb{P}^1 \end{array}$$ is a family of K3 surfaces,

over $\lambda \in T(\alpha)$. the fiber is: $\underset{aI+bJ+cK}{(S, \underset{\parallel}{\lambda})}$

(S, I) $\underset{\sim}{\alpha}$ Kähler form $\to$ Kähler metric

$\rightsquigarrow$ I, J, K on S.

$\to$ $$\begin{array}{c} \mathcal{X}(\alpha) \\ \downarrow \\ T(\alpha) \ni \lambda. \end{array}$$ fiber $(S, \lambda = aI + bJ + cK)$

called twistor space associated with α.

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take a marking f of S.

f extends to markings of all fibers of $\begin{array}{c} \mathcal{X}(\alpha) \\ \downarrow \\ T(\alpha) \end{array}$.

$$\Rightarrow \mathcal{P}: T(\alpha) \longrightarrow D_{K3}.$$

$$W_\alpha = \langle Re(\sigma), Im(\sigma), \alpha \rangle.$$

$$f(W_\alpha) \subset (\Lambda_{K3})_{\mathbb{R}}.$$

Prop: $\mathcal{P}$ identifies $T(\alpha) \cong \mathbb{P}^1$ with the twistor line $T_{f(W_\alpha)}$

Pf: take a fiber (S,λ) of $\begin{array}{c} \mathcal{X}(\alpha) \\ \downarrow \\ T(\alpha) \end{array}$.

need to calculate $f(\sigma_\lambda)$. $\sigma_\lambda \in H^2(S_\lambda)$.

S_λ: S with complex structure λ.

find $\mu, \rho \in T(\alpha)$, such that $\lambda^2 = \mu^2 = \rho^2 = -1$

$\lambda\mu = \rho$, $\mu\rho = \lambda$, $\rho\lambda = \mu$.

Similar as $\sigma_I = \omega_J + I\,\omega_K$.

we have $\sigma_\lambda = \omega_\mu + \lambda\,\omega_\rho$.

So $f(\sigma_\lambda)$ corresponds to the positive plane

$f\langle \omega_\mu, \omega_\rho \rangle \subset f(W_\alpha)$.

So $\mathcal{P}(T(\alpha)) \subset T_{f(W_\alpha)}$.

$\langle \omega_\mu, \omega_\rho \rangle$ goes over all planes of W_α.

$\Rightarrow$ $\mathcal{P}: T(\alpha) \xrightarrow{\simeq} T_{f(W_\alpha)}$. $\square$

Let M be a compact, Kähler surface.

$C_M \subset H^{1,1}(M) \cap H^2(M, \mathbb{R}) =: H^{1,1}(M, \mathbb{R})$

C_M is the connected component of

$\{ \alpha \in H^{1,1}(M,\mathbb{R}) \mid (\alpha,\alpha) > 0 \}$ containing a Kähler class

C_M: called the positive cone of M.

$K_M \subset C_M$ Kähler cone:

K_M is the set of Kähler classes. [cohomology classes of Kähler forms)

K_M is also defined for Kähler manifold M of any dimension.

Demailly, Paun: M compact Kähler manifold,

the Kähler cone $K_M \subset H^{1,1}(M,\mathbb{R})$ is a connected component of the cone of classes $\alpha \in H^{1,1}(M,\mathbb{R})$ defined by the condition $\int_Z \alpha^d > 0$ for any subvarieties $Z \subset M$ of dimension $d > 0$.

For a complex K3 surface S, if $\mathrm{Pic}(S) = 0$, K_S is a connected component of the set of $\alpha \in H^{1,1}(S,\mathbb{R})$ satisfying $\int \alpha^2 > 0$ (i.e. $\varphi(\alpha,\alpha) > 0$)

$\Rightarrow K_S = C_S$. Kähler cone = positive cone.

If $\mathrm{Pic}(S) = 0$, then any $\alpha \in H^{1,1}(S,\mathbb{R})$ with positive norm, we have either α Kähler or $-\alpha$ Kähler.

generic twistor line.

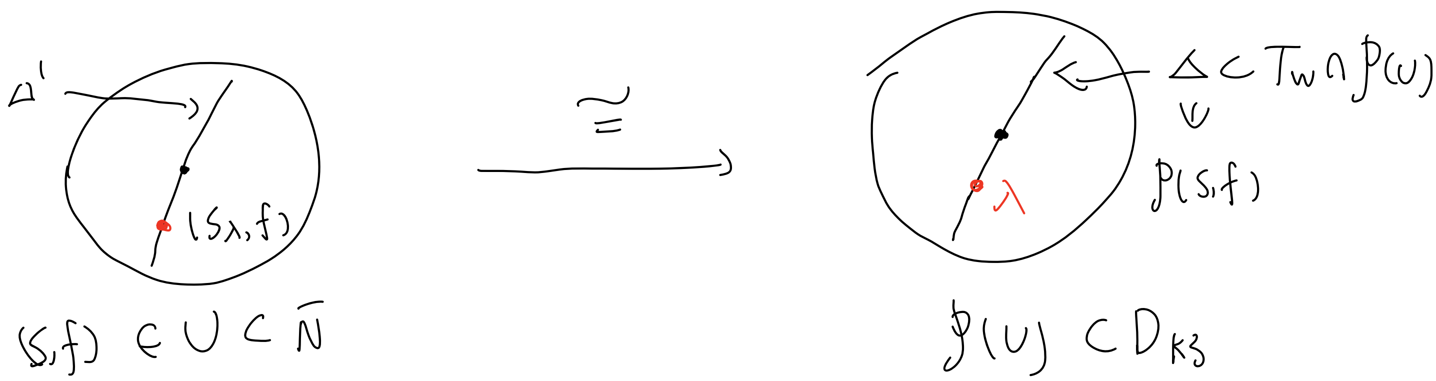
Prop: $(S,f) \in N$ marked K3, assume $p(S,f) \in D_{K3}$ is contained in a generic twistor line $Tw \subset D$. Then there exists a unique lifting of Tw to a curve in $\bar{N}$ through (S,f).

$$\begin{tikzcd} \bar{N} \arrow[r, "p"] & D_{K3} \\ & Tw \arrow[u, hook, "i"'] \arrow[ul, "\tilde{i}"] \arrow[u, phantom, shift right=6, "\circlearrowleft"] \end{tikzcd}$$

$\tilde{i}(Tw) \ni (S,f)$.

$\tilde{i}$ is unique.

Pf: p is locally bihol



$$\Delta' \xrightarrow{1:1} \Delta.$$

Tw is generic $\Rightarrow$ $\exists\ \lambda \in \Delta$ s.t. $\lambda^\perp \cap \Lambda_{K3} = 0$

(S_λ, f) a marked K3 in Δ', with $p(S_\lambda, f) = \lambda$

$f\, H^{2,0}(S_\lambda) = \lambda$

$\Rightarrow \quad Pic(S_\lambda) = 0$

take $\alpha \in W$, s.t. $\alpha \perp Re(\lambda)$, $\alpha \perp Im(\lambda)$,

$\langle \alpha, Re(\lambda), Im(\lambda) \rangle_{\mathbb{R}} = W$

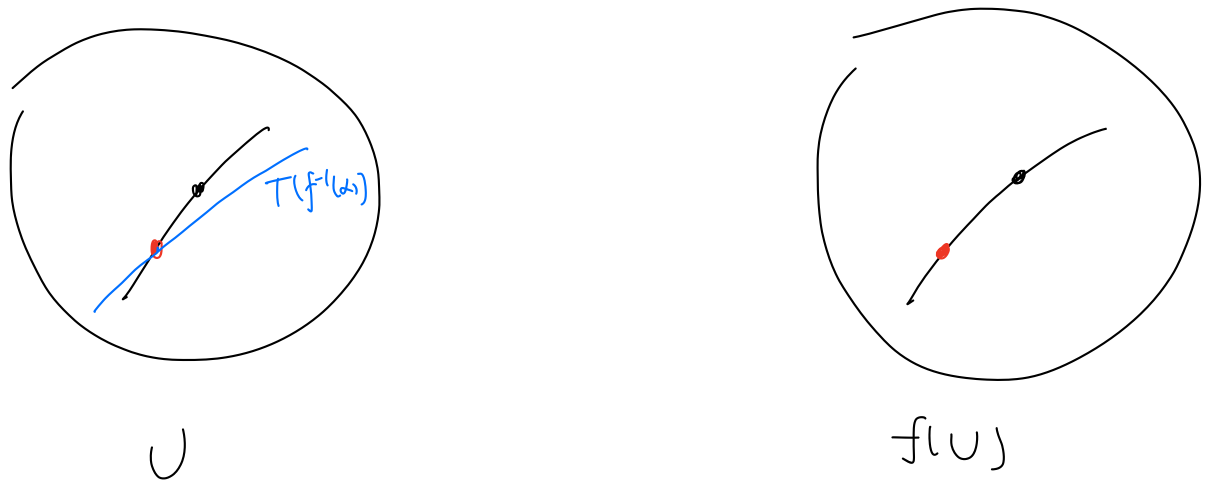
$\alpha \perp \lambda$
$\alpha \perp \bar{\lambda}$ $\Rightarrow \quad f^{-1}(\alpha) \in H^{1,1}(S_\lambda, \mathbb{R})$

We have either $f^{-1}(\alpha)$ Kähler or $-f^{-1}(\alpha)$ Kähler,

WLOG: assume $f^{-1}(\alpha)$ Kähler.

$\mathcal{X}(f^{-1}(\alpha))$ twistor family of K3 surfaces.
$\downarrow$
$T(f^{-1}(\alpha))$

$$p : T(f^{-1}(\alpha)) \xrightarrow{\cong} T_{f(W_{f^{-1}(\alpha)})} = T_{W_\alpha}$$



Lem: $p : \underset{x}{X} \to \underset{y}{Y}$ is a continuous map between two Hausdorff topological manifolds, and p is locally homeomorphism, then any continuous map

$\Delta \to Y$ can be lifted uniquely to
$0 \mapsto y$

$\Delta \to X$ with $0 \mapsto x$.