Lecture 17 11 May 2022

Thm (Γ, r) and $u \in D$ ($u : V \to \mathbb{R}$, with finite energy).
Then for almost every one-sided path p in Γ,
$u(x)$ converges as x tends to ∞ along p.

Pf Pick $a \in V$ and P_a = set of all one-sided paths starting at a. $\overset{\overset{a}{\shortparallel}}{(x_0}, x_1, x_2, \ldots)$

Define $P \subset P_a$ which contains all paths $\overset{\overset{\shortparallel}{}}{p} \in P_a$ s.t.

$$(*) \quad \sum_{j=1}^{\infty} |u(x_j) - u(x_{j-1})| < \infty .$$

Idea: $\mathcal{P}_a = (\mathcal{P}) \cup (\mathcal{P}_a - \mathcal{P})$

Claim: ① $\lim u$ exists for all paths in $\mathcal{P}$.

② $\lambda(\mathcal{P}_a - \mathcal{P}) = \infty$

Proof of ①:

$$u(x_n) - u(a) = \sum_{j=1}^{n} u(x_j) - u(x_{j-1})$$

Along $p \in \mathcal{P}$, $(*) \Rightarrow \{u(x_n)\}$ is Cauchy-sequence

$\Rightarrow$ limit exists.

Proof of ②: Consider I 1-chain given by

$$I(xy) = c(xy)\big(u(x) - u(y)\big).$$

$$W(I) = D(u) < \infty .$$

For any $p \in \mathcal{P}_a - \mathcal{P}$

$$L_I(p) = \sum_{xy \in \bar{\partial}(p)} r(xy) \left| I(x,y) \right|$$

$$= \sum_{xy \in \bar{\partial}(p)} \cancel{r(xy)} \cdot \cancel{c(xy)} \left| u(x) - u(y) \right|$$

$$\to \infty ,$$

$$\Rightarrow \quad L_I(\mathcal{P}_a - \mathcal{P}) = \inf_{p \in \mathcal{P}_a - \mathcal{P}} L_I(p) = \infty .$$

$$\lambda(\mathcal{P}_a - \mathcal{P}) = \sup_{I \text{ nontrivial}} \frac{L_I^2(\mathcal{P}_a - \mathcal{P})}{W(I)} \geq \frac{L_I^2(\mathcal{P}_a - \mathcal{P})}{W(I)} = \infty .$$

Consider $\bigcup_{a \in V} P_a$ set of all one sided paths

$\overline{P}$ set of one-sided paths where the limit does not converge

$$\frac{1}{\lambda(\overline{P})} \leq \sum_{a \in V} \frac{1}{\lambda(P_a \cap \overline{P})} = 0$$

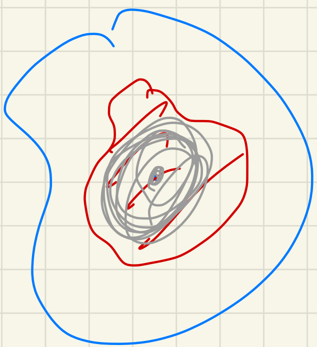
$$\Rightarrow \lambda(\overline{P}) = \infty .$$

Thm (Γ, r), $u \in D_0$ $\left(\exists\ u_n : V \to \mathbb{R}\right.$ with finite support s.t. $\left.\lim_{n\to\infty} \| u_n - u \| = 0 \right)$

Then for every $a \in V$, almost every path in P_a we have

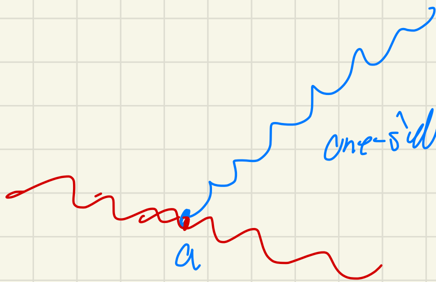
$$\lim u(x) = 0$$

as x tends to ∞ along p.



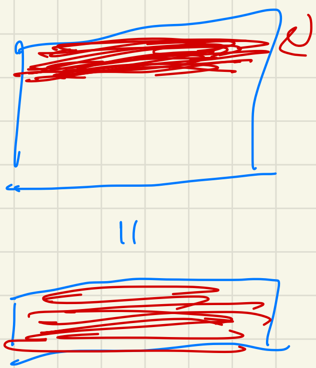
$$\left(\| u_k - u \| = \sqrt{(u_k(0) - u(0))^2 + D(u_k - u)} \right)$$

~~Remark:~~ Exercise: If P contains infinite paths staying in a compact set, then $\lambda(P) = \infty$.



one-sided $(x_0, x_1, x_2 \ldots \ldots x_i \ldots) = \mathbb{Z}_+$

two-sided $(\quad x_{-1}, x_0, x_1 \quad) = \mathbb{Z}$



$$\frac{1}{\lambda(P)} = \inf_{\substack{\sigma \\ L_\sigma(P)=1}} \operatorname{Area}(\sigma)$$

$$\lambda(P) = \sup_{\substack{\sigma \\ \operatorname{Area}(\sigma)=1}} L_\sigma(P).$$

$$\lambda(P) < \infty.$$

$\lambda(P)$ ignore the "hole"

Thm (Γ, r) locally finite transient. Let $u \in HD$.

If $\exists$ const. c s.t.

$$\lim u(x) = c$$

along the vertices of almost every one-sided path.

$$\Rightarrow \quad u(x) = c \quad \forall x \in V.$$

Corollary, ~~Thm~~

$f \in D$ belongs to D_0

$\iff \lim f(x) = 0$ as $x \to \infty$ along almost every one-sided paths,

Pf: $f = (u)_{HD} + (v)_{D_0}$

Remarks:

Z collection of all $\overset{\text{finite,}}{\downarrow}$ cycles

i.e. $I \in Z \iff \partial I = 0$

$\iff \sum_y I(x,y) = 0 \quad \forall x \in V,$

$Z \subset Z^*$ collection of all cycles, infinite or finite,

Lemma:

$$(*)\begin{cases} \partial I = 0 & \forall x \in V \\ \langle I, z \rangle = 0 & \forall \text{ finite cycles } z \end{cases}$$

has only trivial solutions $\iff Z = Z^*$

$\iff$ recurrent

Pf: If $Z^* \neq Z \Rightarrow I \in Z^*$ and $I \perp Z$. non-trivial

$\Rightarrow I$ solves $(\maltese)$

<u>Thm</u> $(\maltese)$ has trivial solns in H_1

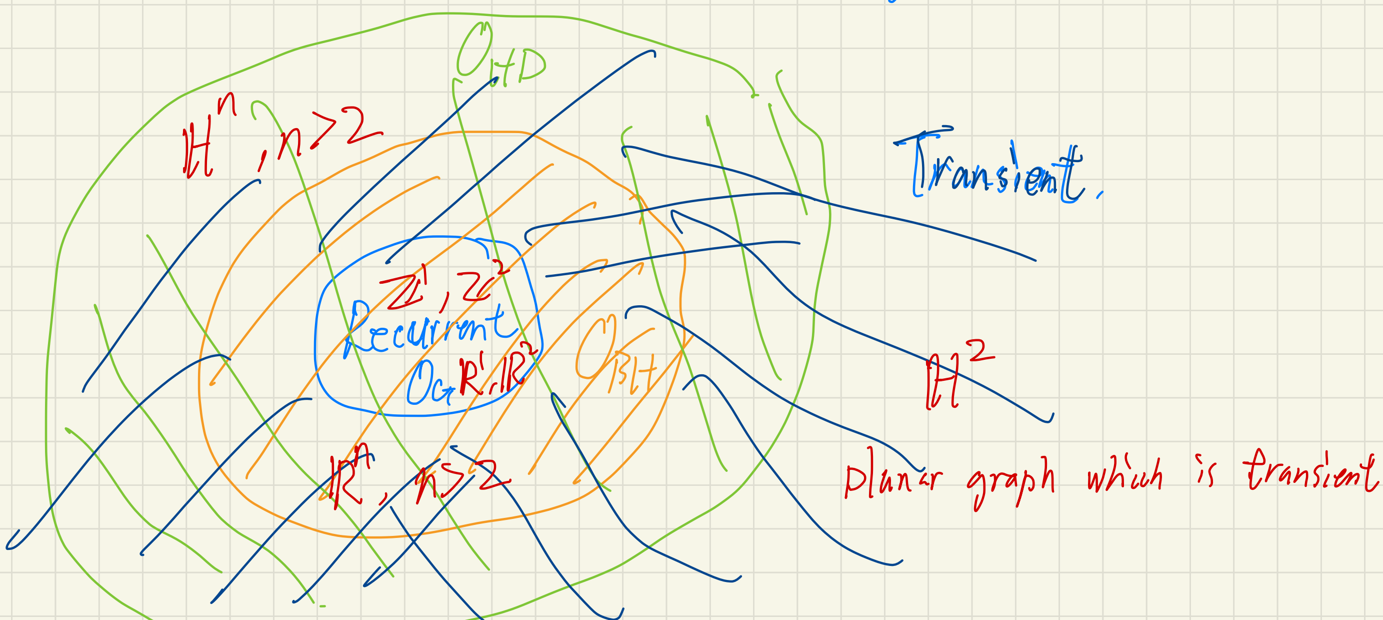
$\Leftrightarrow$ for any $a, b \in V$, the limit current and the minimal current generated $\eta := \delta_a - \delta_b$ coincides.

<u>Def</u>. $\mathcal{O}_G$ collection (Γ, r) recurrent.

$\mathcal{O}_{BH}$ collection of (Γ, r) s.t. every bounded harmonic function is constant.

O_{HD} . . . . s.t. every harmonic Dirichlet function is constant.

Classification of electric networks / Riemannian mfds.



(Γ, r) is said to have weak-Liouville property

$$O_G \subset O_{BH} \subset O_{HD}$$

Recall: ① Recurrent $\Rightarrow$ positive superharmonic is const.

$\Rightarrow$ harmonic Dirichlet function is constant.

$$\Rightarrow O_G \subset O_{HD}$$

② every $u \in HD$ can be approximated by a sequence of bounded harmonic Dirichlet function

$$\Rightarrow (\Gamma, r) \notin O_{HD} \Rightarrow (\Gamma, r) \notin O_{BH}.$$

$$\Rightarrow \quad \mathcal{O}_{BH} \subset \mathcal{O}_{HD},$$