complex K3 is simply connected.

twistor line, prove the surjectivity of the local period map for marked K3 surfaces.

$N :=$ set of isomorphism classes of marked complex K3 surfaces

N has a natural structure as a (non-Hausdorff) complex manifold of dimension 20.

Rmk: N is not connected. Actually N has two connected components,

For example, take a marked complex K3 surface (S, f).

$$f : H^2(S, \mathbb{Z}) \longrightarrow \Lambda_{K3}$$

$-f$ is also a marking of S,

Actually, (S, f), $(S, -f)$ lie in different connected components of N.

Any complex K3 admits a unobstructed universal deformation

$$S \rightsquigarrow \begin{array}{c} \mathcal{S} \\ \downarrow \\ \mathrm{Def}(S) \end{array}$$ $\mathrm{Def}(S)$ is smooth.

(S, f), then f can be extended to be marking for all fibers of $\begin{array}{c}\mathcal{S}\\ \downarrow\\ \mathrm{Def}(S)\end{array}$.

$\mathrm{Def}(S) \hookrightarrow N$. open subset of N.

All such $\mathrm{Def}(S)$ glue to N, and give rise to the complex mfd structure on N.

S_1, S_2 two complex K3.
f_1 f_2

$\mathrm{Def}(S_1) \cap \mathrm{Def}(S_2) \neq \phi$

$\begin{array}{c}\mathcal{S}_1\\ \downarrow\\ \mathrm{Def}(S_1)\end{array}$ f_1 $\begin{array}{c}\mathcal{S}_2\\ \downarrow\\ \mathrm{Def}(S_2)\end{array}$ f_2,

There is a unique way to identify the restrictions of (S_1, f_1), (S_2, f_2) to $\mathrm{Def}(S_1) \cap \mathrm{Def}(S_2)$.

[Lem: The automorphism group of a complex K3 surface acts faithfully on H^2;

equivalently, if $g: S \to S$ biholomorphic,

and $g^*: H^2(S, \mathbb{Z}) \to H^2(S, \mathbb{Z})$ is trivial,

then g is the identify].

We have a family of marked complex K3 over N

This is the "universal" family of marked K3.

N is the "fine" moduli space of marked complex K3.

$\Lambda_{K3} \cong U^3 \oplus E_8(-1)^2$. $\quad \varphi : \Lambda_{K3} \times \Lambda_{K3} \to \mathbb{Z}$,

$$D_{K3} = \mathbb{P}\{ x \in (\Lambda_{K3})_{\mathbb{C}} \mid \varphi(x,x) = 0, \ \varphi(x, \bar{x}) > 0 \}.$$

$= $ set of oriented positive planes in $(\Lambda_{K3})_{\mathbb{R}}$.

local period map : $\mathcal{P} : N \longrightarrow D_{K3}$.

$$\mathcal{P}(S, f) = f[H^{2,0}(S)].$$

$d\mathcal{P}$ is an isomorphism at every point of N.

$$\begin{tikzcd}
d\mathcal{P} : T_{(S,f)} N \arrow[r, "\cong"] \arrow[d, equal] & T_{\mathcal{P}(S,f)} D_{K3} & \\
T_{[S]} \mathrm{Def}(S) \arrow[r, "K.S."'] & H^1(S, T_S) \arrow[r, "\cong"] & H^1(S, \Omega_S) \arrow[ul, "\cong"']
\end{tikzcd}$$

(Local Torelli): $\mathcal{P}$ is a local isomorphism.

i.e. $\forall (S,f) \in N$, $\exists$ open nbhd $B \ni (S,f)$,

$\mathcal{P} : B \to \mathcal{P}(B)$ is biholomorphic.

Take a complex K3 S, a marking f,

Apriori we only know $H^{0,1} = H^{1,0} = H^1 = 0$.

$$N \supset B \ni (S,f) \qquad \mathcal{P} : B \underset{\text{biholomorphic}}{\xrightarrow{\cong}} \mathcal{P}(B) \subset D_{K3}.$$

Lem (lattice theory): for any $0 \neq \alpha \in \Lambda_{K3}$, the set

$$\bigcup_{g \in \mathrm{Aut}(\Lambda_{K3})} g(\alpha)^{\perp} \text{ is dense in } D_{K3}.$$

(for proof, see Huybrechts' book, page 129).

$g(\alpha) \in \Lambda_{K3}$, $\quad g(\alpha)^{\perp} := \{ [x] \in D_{K3} \mid \varphi(g(\alpha), x) = 0 \}$,

19-dim'l submanifold of D_{K3}.

($\mathrm{Aut}(\Lambda_{K3})$ has enough elements to make $\cup\, g(\alpha)^{\perp}$ dense)

Choose α a primitive element of norm 4 in Λ_{K3}.

[Actually, such elements form a single $\mathrm{Aut}(\Lambda_{K3})$-orbit].

Then $\exists\ g \in \mathrm{Aut}(\Lambda_{K3})$, such $g(\alpha)^{\perp} \cap \mathcal{P}(B) \neq \phi$.

$\Rightarrow \exists\ (S_1, f_1) \in B$, s.t. $\mathcal{P}(S_1, f_1)$ is a generic element in $g(\alpha)^{\perp}$. $\Rightarrow f_1 (H^{2,0}(S_1))^{\perp} \cap \Lambda_{K3} = \langle g(\alpha) \rangle$.

$\overset{\shortparallel}{f_1 [H^{2,0}(S_1)]} \in D_{K3}$.

$\Rightarrow S_1$ is complex K3 surface with $\mathrm{Pic}(S_1)$ generated by $f_1^{-1}(g(\alpha))$ [a primitive vector of norm 4].

$\Rightarrow f_1^{-1}(g(\alpha))$ is very ample and realizes S_1 as a smooth quartic surface in $\mathbb{P}^3$.

Every complex K3 is deformation equivalent to certain smooth quartic surface

Facts: Each two smooth quartic surfaces are deformation equivalent.

By weak Lefschetz, a smooth quartic surface is simply connected.

Conclusion: Each two complex K3 surfaces are deformation equiv. and every complex K3 is simply connected.

Next we discuss $\mathcal{P}: N \to D_{K3}$.

Prop: $\mathcal{P}$ factorizes uniquely over a Hausdorff space $\bar{N}$; i.e. $\exists!$ complex Hausdorff manifold $\bar{N}$ with

$$\mathcal{P}: N \twoheadrightarrow \bar{N} \xrightarrow[\text{locally biholomorphic}]{} D_{K3}$$

such that two (S_1, f_1), $(S_2, f_2) \in N$ map to the same point in $\bar{N}$ iff they are inseparable in N.

[Any open nbhds of (S_1, f_1), (S_2, f_2) intersect].

Verbitsky's proof of the global Torelli (and surjectivity) for hyper-Kähler mfd. also need this prop.

Aim:

Thm: $\bar{N}$ has two connected components, each component is mapped biholomorphically to D_{K3}.

The first step: show surjectivity

Twistor line.

Let Λ be a lattice of signature $(3, b-3)$

(for $\Lambda = \Lambda_{K3}$, $b = 22$). $\varphi : \Lambda \times \Lambda \to \mathbb{Z}$.

$\Lambda_{\mathbb{C}} \times \Lambda_{\mathbb{C}} \to \mathbb{C}$.

$W < \Lambda_{\mathbb{R}}$ $(= \Lambda \otimes_{\mathbb{Z}} \mathbb{R})$ positive 3-subspace.

i.e. $\varphi : W \times W \to \mathbb{R}$ is positive,

$D = \mathbb{P}\{ x \in \Lambda_{\mathbb{C}} \mid \varphi(x,x) = 0,\ \varphi(x, \bar{x}) > 0 \}$

$=$ set of oriented positive planes in $\Lambda_{\mathbb{R}}$.

[D is simply-connected].

$W \rightsquigarrow$ consider $T_W = D \cap \mathbb{P}(W_{\mathbb{C}})$.

Any $x \in W_{\mathbb{C}} - 0$ satisfies $\varphi(x, \bar{x}) > 0$ because $W_{\mathbb{R}}$ is pos.

So $T_W = \mathbb{P}\{ x \in W_{\mathbb{C}} \mid \varphi(x,x) = 0 \}$.

is a smooth quadric curve in $\mathbb{P} W_{\mathbb{C}} \cong \mathbb{P}^2$.

T_W is isomorphic to $\mathbb{P}^1$.

(Veronese embedding $\mathbb{P}^n \hookrightarrow \mathbb{P}^N$. $N = \frac{1}{2}(n+1)(n+2) - 1$.

$[x_0 : \cdots : x_n] \mapsto [x_i x_j]_{0 \le i \le j \le n}$

$n = 1$. $N = 2$.).

T_W is called a twistor line in D.

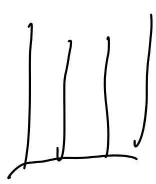
$$T_W = \mathbb{P}W_{\mathbb{C}} \cap D \hookrightarrow \mathbb{P}W_{\mathbb{C}}$$
$$\searrow D.$$

Call T_W $[W]$ generic, if $\exists\ x \in T_W$ such that $x^{\perp} \cap \Lambda = 0$

$\Longleftrightarrow \exists\ w \in W$, s.t. $w^{\perp} \cap \Lambda = 0$

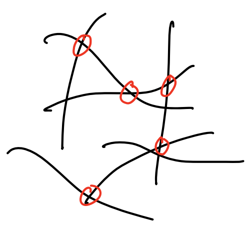
If W is generic, then $x^{\perp} \cap \Lambda = 0$ for all but finitely many $x \in T_W$.

$$p : N \longrightarrow \bar{N} \longrightarrow D_{K3}.$$

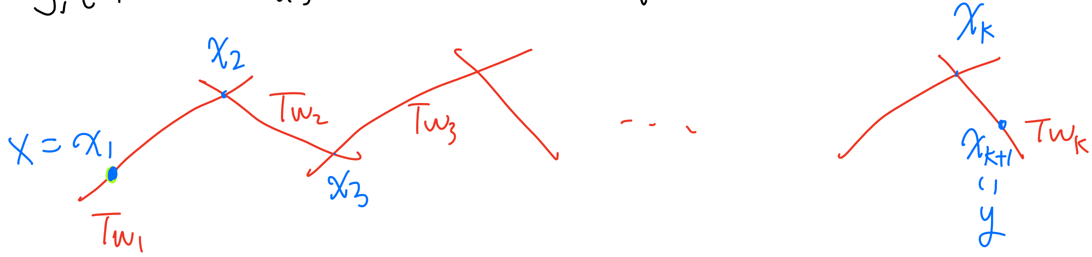


$\mathbb{P}^1$-family of K3 surfaces.

from Hyper-Kähler structure.



Defn. Two points $x, y \in D$ are called equivalent, if $\exists$ a chain of generic twistor lines $T_{W_1}, \cdots, T_{W_k}$, and points $x = x_1, x_2, \cdots, x_k, x_{k+1} = y$, s.t. $x_i, x_{i+1} \in T_{W_i}$.



Prop: Any two points $x, y \in D$ are equivalent.

Pf.: $x = a+bi$, $a, b \in \Lambda_{\mathbb{R}}$,

$\langle a, b\rangle < \Lambda_{\mathbb{R}}$ positive plane,

take $c \in \Lambda_{\mathbb{R}}$, s.t. $\langle a, b, c\rangle = W$ positive three-subspace
$c^{\perp} \cap \Lambda = 0$.

$\exists$ open nbhd U of a, V of b, s.t.

$\forall$ $a' \in U$, $b' \in V$, we have $\langle a', b', c\rangle$ is a positive 3-subspace.

$\langle a, b, c\rangle = W_1$ $\quad \langle a, b', c\rangle = W_2$, $\quad \langle a', b', c\rangle = W_3$.

$$\langle a, b\rangle \xleftrightarrow[T_{W_1}]{} \langle a, c\rangle \xleftrightarrow[T_{W_2}]{} \langle b', c\rangle \xleftrightarrow[T_{W_3}]{} \langle a', b'\rangle.$$

W_1, W_2, W_3 are all generic since $c^{\perp} \cap \Lambda = 0$,

So $\langle a, b\rangle \sim \langle a', b'\rangle$ for any $a' \in U$, $b' \in V$.

So any equivalence orbit is open.

$\Rightarrow D$ is a union of open orbits,

D is connected $\Rightarrow$ D cannot be a disjoint union of more than one nonempty open subsets.

$\Rightarrow$ D has only one orbit

$\Rightarrow$ Any two elements $x, y \in D$ are equivalent. $\square$