Hopf algebras

In this section, we will see that, regarding to Reconstruction Theorem, the difference between bialgebra and Hopf algebra, is the same as ———— finite ring cat. and finite tensor cat.

(we will also mention w/o finite later, i.e. infinite setting)

At the algebra level, the difference is given by an invertible antipode.

—— cat. ———————— rigid structure.

In this section, we will mention a lot of results without proof (see Section 5.3 for the details).

Consider $H = \mathrm{End}(F)$ as above, and assume that $\mathcal{C}$ has left duals.

Then, we can define a linear map $S : H \to H$, $a \mapsto S(a)$ as follows:

$$S(a)_X := \left(a_{X^*}\right)^*$$

(where we assume $F(X)^* = F(X^*)$) (without loss of generality...)

<u>Prop</u> (antipode axiom). Let $\mu : H \otimes H \to H$, $i : \Bbbk \to H$ be the multiplication and unit maps. Then:

$$\mu \circ (\mathrm{id} \otimes S) \circ \Delta = i \circ \varepsilon = \mu \circ (S \otimes \mathrm{id}) \circ \Delta.$$

(as maps $H \to H$)

<u>Def:</u> An antipode on a bialgebra H is a linear map $S : H \to H$ satisfying above equalities

The following prop. is the "linear alg" analog of "uniqueness of left dual when it exists (up to unique iso.)"

[It can proved using a reconstruction thm, we will see the finite dim. case soon and the ∞-case later]

Prop: An antipode on a bialgebra H is unique if it exists.

Next:

Prop: Let S be an antipode on a bialgebra H. Then S is an antihomomorphism of algebra: $S(ab) = S(b)S(a)$.

(Idem, of coalgebra)

I guess, it corresponds to $(X \otimes Y)^* \cong Y^* \otimes X^*$.

converse of Prop (antipode exists)

Corollary: (i) If H is a bialgebra with an antipode S, then $\mathcal{C} = \mathrm{Rep}(H)$ has left duals, i.e. $\forall X \in \mathcal{C}$, the left dual X^* is the usual dual space of X, with action on H as follows:

let $a \in H$.

$$\boxed{\ell_{X^*}(a) = \ell_X(S(a))^*}$$ matrix adjoint.

and usual (co)ev maps of Vec.

(ii) If in addition S is invertible, then $\mathcal{C}$ has also right duals (i.e. rigid, $\mathcal{C}$ is tensor cat.), i.e. $\forall X \in \mathcal{C}$, the right dual ${}^*X$ is the usual dual space of X, with action on H, by:

(let $a \in H$)

$$\boxed{\ell_{{}^*X}(a) = \ell_X(S^{-1}(a))^*}$$

and usual (co)ev maps of Vec.

<u>Rk:</u> Idem for H-Comod instead of Rep(H)

Definition: A bialgebra with an invertible antipode is called **Hopf algebra**.

Reconstruction Theorem: the assignments $(\mathcal{C}, F) \mapsto H = \mathrm{End}(F)$
$H \mapsto (\mathrm{Rep}(H), \mathrm{Forget})$

are mutually inverse bijection between:

(1) equiv. classes of finite tensor cat. $\mathcal{C}$ with a fiber functor F (up to ... as before)

(2) iso classes of finite dim. Hopf algebras over $\mathbb{k}$.

Rk: (not in the book) . We "should" have such a reconstruction thm for unitary fusion cat. (over $\mathbb{C}$) and finite dim. Hopf $*$-alg over $\mathbb{C}$ (sometimes called Kac algebra)
with (unitary?) fiber functor F.

(and also depth 2 irreducible + finite index subfactor planar algebra (over $\mathbb{C}$)

Now, "lin. alg" analog of "a finite ring cat. $\mathcal{C}$ with left duals is a tensor cat."
(i.e. it also has right duals)

Prop: If H is a finite dim bialgebra with antipode S, then S is invertible, so H is Hopf algebra.

proof: by reconstruction thm ... (the book provides an alg. proof) $\boxtimes$

Notations / Exo: Let $H = (H, \mu, i, \Delta, \varepsilon, S)$ be a Hopf algebra over a field k. The following are again Hopf algebras:

- $H_{op} := (H, \mu^{op}, i, \Delta, \varepsilon, S^{-1})$ (opposite)
- $H^{cop} := (H, \mu, i, \Delta^{op}, \varepsilon, S^{-1})$
- $H_{op}^{cop} := (H, \mu^{op}, i, \Delta^{op}, \varepsilon, S)$
- $H^* := (H^*, \Delta^*, \varepsilon^*, \mu^*, i^*, S^*)$ If H is finite dim

$\mu^{op}(a \otimes b) = \mu(b \otimes a)$

If $\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)}$

then $\Delta^{op}(a) = \sum a_i^{(2)} \otimes a_i^{(1)}$.

So: $\mu \circ \Delta = \mu^{op} \circ \Delta^{op}$.

Exo: Let $(H_r, \mu_r, i_r, \Delta_r, \varepsilon_r, S_r)$ $r = 1, 2$ by two Hopf algebras. Find the Hopf algebra structure on $H_1 \otimes H_2$. (i.e. find $\mu, i, \Delta, \varepsilon, S$)

Reconstruction theory in the infinite setting

Rk: Here we will need to use $Coend(F)$ instead of $End(F)$ because (I guess) if $\mathcal{C}$ (and H-comod ——— $Rep(H)$). has left duals then $Coend(F)$ has antipode (but not(?) $End(F)$ in general)

Let $\mathcal{C}$ be a ring category (over k), C is not assumed finite. Let F be a fiber functor. We will need to redefine the set $End(F)$ (of natural transformations $\phi: F \to F$, given by series $(\phi_X)_{X \in \mathcal{C}}$ + commuting diagr.)

as an "inverse limit" $\varprojlim \mathrm{End}(F(X))$ (we will recall it soon)

(projective limit)

Then, the dual $\varinjlim$ (denoted direct limit or inductive limit)

will be used to define $\mathrm{Coend}(F)$ as $\varinjlim \mathrm{End}(F(X))^*$

Idea about projective limit and inductive limit:

$\varprojlim$ provides a subset of a product $\prod$

$\varinjlim$ ——— quotient of a coproduct $\coprod$ (disjoint union in Set)

Projective limit: Let $\mathcal{C}, \mathcal{D}$ be two locally small (abelian) cat.

$G : \mathcal{C} \to \mathcal{D}$ (faithful) functor, $i \mapsto X_i$, $i \xrightarrow{t_{ij}} j \mapsto X_i \xrightarrow{t_{ij}} X_j$

The object of $\mathcal{C}$ will be denoted by i (for index)

—— $\mathcal{D}$ —————— X_i (for $G(i)$)

element of $\hom_{\mathcal{C}}(i,j)$ —— ι_{ij}

————— $\hom_{\mathcal{D}}(X_i, X_j)$ —— f_{ij} ($f_{ij} = G(\iota_{ij})$)

Warning about notation: for our need

$\mathcal{G}$ = rig cat with fiber functor f

$\mathcal{D}$ = Set (or Vec or Mat)

$G = \mathrm{End}(f(i))$

$\mathrm{End}(f(X))$ = matrix alg

first assume $\mathcal{D}$ to be Set. Then

$$\varprojlim X_i = \left\{ (x_i) \in \prod_{i\in C} X_i \text{ s.t. } \forall f_{ij} \,,\ f_{ij}(x_i) = x_j \right\} \subset \prod X_i$$

it can be redefine by the following universal property:

$\forall Y \in D \quad \exists!\, \psi$ s.t. $\forall i \in C$, $\forall \varphi_i$, $\forall f_{ij}$

$\exists$ projection π_i the following diag. commutes

$$\begin{tikzcd}
 & & X_i \\
Y \arrow[urr, "\varphi_i"] \arrow[r, dashed, "\psi"] \arrow[drr, "\varphi_j"'] & \varprojlim X_k \arrow[ur, dashed, "\pi_i"'] \arrow[dr, "\pi_j"] & \\
 & & X_j
\end{tikzcd}$$

in other words:

$$\varprojlim \hom_D(Y, X_i) \simeq \hom_D(Y, \varprojlim X_i)$$

as above

by this def, D can be not a set but....

See you later --- for more details →