Period map for K3 surfaces.

S: complex K3 surface.

$H^2(S,\mathbb{Z})$ is even unimodular lattice of signature $(3,19)$.

By Milnor's classification:

$$H^2(S,\mathbb{Z}) \cong U^3 \oplus E_8(-1)^2.$$

We have Hodge decomposition:

$$H^2(S,\mathbb{C}) = \underset{1}{H^{2,0}} \oplus \underset{20}{H^{1,1}} \oplus \underset{1}{H^{0,2}}.$$

φ: $H^2(S,\mathbb{C}) \times H^2(S,\mathbb{C}) \longrightarrow \mathbb{C}$, $(x,y) \mapsto \int_S x \wedge y$.

φ: $H^2(S,\mathbb{Z}) \times H^2(S,\mathbb{Z}) \longrightarrow \mathbb{Z}$.

$\forall\ \omega \in H^{2,0} - 0$ a nowhere vanishing holomorphic 2-form on S

$$\Rightarrow \int_S \omega \wedge \bar{\omega} > 0, \quad \int_S \omega \wedge \omega = 0.$$

Now denote by $\Lambda_{K3} = U^3 \oplus E_8(-1)^2$.

Call it K3 lattice.

For any complex K3 surface S, there exist isomorphisms

$$f: H^2(S,\mathbb{Z}) \xrightarrow{\cong} \Lambda_{K3}.$$

Call (S,f) a marked K3 surface,

f called a marking of S.

Given a marked K3 surface (S, f),

can look at $f\left(H^{2,0}(S)\right) \subset \Lambda_{K3} \otimes \mathbb{C}$

It satisfies $\varphi(w, \bar{w}) > 0$, $\varphi(w,w)=0$. $\varphi: \Lambda_{K3} \times \Lambda_{K3} \to \mathbb{Z}$

for $w \in f(H^{2,0}(S) - 0)$

Define $D_{K3} = \{ \ell \subset \Lambda_{K3} \otimes \mathbb{C} \mid \dim_{\mathbb{C}} \ell = 1,\ \varphi(w,w)=0,\ \varphi(w, \bar{w}) > 0$ for $w \in \ell - 0 \}$,

$$= \mathbb{P}\{ w \in \Lambda_{K3} \otimes \mathbb{C} - 0 \mid \varphi(w, \bar{w}) > 0,\ \varphi(w,w) = 0 \}.$$

$$\subset \mathbb{P}(\Lambda_{K3})_{\mathbb{C}}$$

$(\Lambda_{K3})_{\mathbb{C}} = \Lambda_{K3} \otimes \mathbb{C}$ has dimension 22.

$\mathbb{P}(\Lambda_{K3})_{\mathbb{C}} \cong \mathbb{P}^{21}$.

$\mathbb{P}\{ w \in (\Lambda_{K3})_{\mathbb{C}} - 0 \mid \varphi(w,w) = 0 \}$ is a quadric hypersurface

$\dim = 20$.

$D_{K3} = \mathbb{P}\{ w \in (\Lambda_{K3})_{\mathbb{C}} - 0 \mid \varphi(w, \bar{w}) > 0,\ \varphi(w,w) = 0 \}$

is an open subset of the quadric hypersurface

D_{K3} is called the local period domain for complex K3 surfaces.

It is a non-compact complex manifold of dimension 20.

Fact: D_{K3} is connected.

D_{K3} can be regarded as the set of oriented positive planes in $(\Lambda_{K3})_{\mathbb{R}}$.

$[w] \in D_{K3}$. $\quad \varphi(w,w) = 0 \quad \varphi(w,\bar{w}) > 0$

$w \in (\Lambda_{K3})_{\mathbb{C}} = (\Lambda_{K3})_{\mathbb{R}} + i\,(\Lambda_{K3})_{\mathbb{R}}$.

$w = a + bi, \quad a, b \in (\Lambda_{K3})_{\mathbb{R}}$.

$$0 = \varphi(w,w) = \varphi(a+bi,\, a+bi) = \varphi(a,a) + 2i\,\varphi(a,b) - \varphi(b,b)$$

$$\Rightarrow \begin{cases} \varphi(a,b) = 0 \\ \varphi(a,a) = \varphi(b,b) \end{cases}$$

$$0 < \varphi(w,\bar{w}) = \varphi(a+bi,\, a-bi) = \varphi(a,a) + \varphi(b,b).$$

So condition on $w \Leftrightarrow \begin{cases} \varphi(a,b) = 0 \\ \varphi(a,a) = \varphi(b,b) > 0. \end{cases}$

$\langle a, b\rangle \subset (\Lambda_{K3})_{\mathbb{R}}$.

fix order   oriented plane

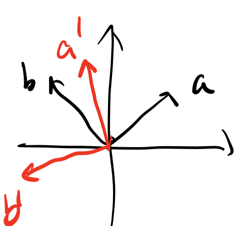
$$\varphi(\lambda a + \mu b,\, \lambda a + \mu b) = \lambda^2 \varphi(a,a) + \mu^2 \varphi(b,b) \geq 0$$

equality holds $\Leftrightarrow \lambda = \mu = 0$.

$\langle a, b\rangle$ is an oriented positive plane in $(\Lambda_{K3})_{\mathbb{R}}$.

Conversely, given such a plane, we can choose a basis a, b, such that $\varphi(a,a) = \varphi(b,b) > 0$, $\varphi(a,b) = 0$.

(a,b) compatible with orientation.



recover $a + bi \in (\Lambda_{K3})_{\mathbb{C}} - 0$.

(a', b') another choice, then

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = e^{\lambda} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \quad \lambda, \theta \in \mathbb{R}.$$

$$a' + b' i = (a + bi) \cdot e^{\lambda + i\theta}$$

$$D_{K3} = \mathbb{P}\{ w \in (\Lambda_{K3})_{\mathbb{C}} - 0 \mid \varphi(w, w) = 0, \ \varphi(w, \bar{w}) > 0 \}$$

= set of oriented positive planes in $(\Lambda_{K3})_{\mathbb{R}}$.

$\varphi: (\Lambda_{K3})_{\mathbb{R}} \times (\Lambda_{K3})_{\mathbb{R}} \longrightarrow \mathbb{R}$ has signature $(3, 19)$

$(S, f) \rightsquigarrow$ a point in D_{K3}.

$f(H^{2,0}(S)) \in$

Call two marked K3 surfaces (S_1, f_1), (S_2, f_2) equivalent

If $\exists$ biholomorphic map $S_1 \xrightarrow[g]{\cong} S_2$, such that:

$[\, H^2(S_1, \mathbb{Z}) \xleftarrow[g^*]{\cong} H^2(S_2, \mathbb{Z}) \,]$

$$\begin{tikzcd} H^2(S_1, \mathbb{Z}) \arrow[r, "f_1"] & \Lambda_{K3} \\ H^2(S_2, \mathbb{Z}) \arrow[u, "g^*"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

is commutative.

$[$ i.e. $f_2 = f_1 \circ g^*]$.

Denote by $\mathcal{M}$ the set of equivalence classes of marked K3 surfaces, then we have a map:

$\wp : \mathcal{M} \longrightarrow D_{K3}$.

called local period map for complex K3 surfaces.

Fact: $\mathcal{M}$ is naturally a non-Hausdorff analytic manifold of dimension 20.

Thm (to be proved later):

$\wp : \mathcal{M} \longrightarrow D_{K3}$ is surjective,

$d\wp$ is bijective at every point in $\mathcal{M}$.

Add polarization:

d: positive even number.

A polarization of a complex K3 surface of degree d is an ample line bundle L on S, such that $L \cdot L = d$, and $[L] \in H^2(S, \mathbb{Z})$ is primitive.

(S, L) called a polarized K3 surface of degree d.

Λ_{K3} contains elements with norm d.

such elements form one $\mathrm{Aut}(\Lambda_{K3})$-orbit.

take $l_d \in \Lambda_{K3}$, $\varphi(l_d, l_d) = d$.

A marking of a polarized K3 surface (S, L) is an isomorphism $f: H^2(S,\mathbb{Z}) \xrightarrow{\cong} \Lambda_{K3}$

$$[L] \longmapsto l_d$$

(S, L, f) : marked polarized K3 surface.

$$f(H^{2,0}(S)) \in D_{K3} = \mathbb{P}\left\{ \omega \in (\Lambda_{K3})_{\mathbb{C}} - 0 \;\middle|\; \begin{array}{l} \varphi(\omega, \omega) = 0 \\ \varphi(\omega, \bar{\omega}) > 0 \end{array} \right\}$$

$\underset{\varphi}{H^{2,0}(S) \perp [L]}$ since $[L] \in H^2(S,\mathbb{Z}) \cap H^{1,1}$.

$\Rightarrow f(H^{2,0}(S)) \perp f[L] = l_d$.

Denote $\Lambda_d = l_d^\perp$: orthogonal complement of l_d in Λ_{K3}.

sublattice of signature $(2, 19)$.

then: $f(H^{2,0}(S)) \in \mathbb{P}\left\{ \omega \in (\Lambda_d)_{\mathbb{C}} \;\middle|\; \begin{array}{l} \varphi(\omega, \omega) = 0 \\ \varphi(\omega, \bar{\omega}) > 0 \end{array} \right\}$

$=: D_d$

local period domain for K3 surface of degree d.

$D_d \subset D_{K3}$ hyperplane in D_{K3}.

$\dim D_d = 19$. $\dim D_{K3} = 20$.

D_d has two connected components. (reason: Λ_d has signature $(2,19)$.)

D_d = set of oriented positive planes in $(\Lambda_d)_{\mathbb{R}}$.

two component $\leftrightarrow$ two orientations.

(S_1, L_1, f_1) and (S_2, L_2, f_2) are called equivalent, if

$\exists\ g: S_1 \xrightarrow{\cong} S_2$, $g^* L_2 = L_1$, $f_2 = f_1 \circ g^*$.

$$\begin{tikzcd} H^2(S_1, \mathbb{Z}) \arrow[r, "f_1"] & \Lambda_{K3} \\ H^2(S_2, \mathbb{Z}) \arrow[u, "g^*"] \arrow[ur, "f_2"'] \arrow[u, phantom, "\mathbb{Z}"] & \end{tikzcd}$$

M_d: set of marked polarized K3 of degree d.

then we have:

$$\mathcal{P}: M_d \longrightarrow D_d.$$

Thm (global Torelli for algebraic K3):

$\mathcal{P}: M_d \to D_d$ is an open embedding.

Deformation theory.

Suppose $\begin{matrix} X_s & X \\ \downarrow & \downarrow \\ s \in & S \end{matrix}$ is a proper map, X, S complex manifold,

submmersion (differential is always surjective)

call this a smooth proper family of complex manifolds X_s

(Ehresmann Thm): If S is connected, then all fibers

X_s are diffeomorphic.

Actually, $s \in S$. $\exists\ s \in U \subset S$ open nbhd.

$$\begin{tikzcd} X_U \arrow[r, "\cong"] \arrow[d] & X_s \times U \arrow[dl] \\ U & \end{tikzcd}$$ diffeomorphism.

For $\begin{matrix} X \\ \downarrow \\ S \end{matrix}$ a smooth proper family,

$\forall s_1, s_2 \in S$, call X_{s_1}, X_{s_2} are deformation equivalent.

call $\begin{matrix} X \\ \downarrow \\ S \end{matrix}$ is a deformation of X_s for $\forall\ s \in S$.

S can be shrinked, and the shrinked family is still deformation of X_s.

$\begin{matrix} X \\ \downarrow \\ S \end{matrix}$ is sometimes understood as a family over a germ around s.

Defn: [S is allowed to be singular, non-reduced].

For X_0, a deformation $\begin{matrix} X \\ \downarrow \\ S \end{matrix}$ is called complete,

if any other deformation $\begin{matrix} X' \\ \downarrow \\ S' \end{matrix}$ of X_0 is a

pullback of $\begin{matrix} X \\ \downarrow \\ S \end{matrix}$.

i.e.
$$\begin{tikzcd} X' = f^{-1}X \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

$S' \xrightarrow{f} S$

A complete deformation is called universal, if the pullback f is always unique.

Called versal, if $df: T_{s'}S' \longrightarrow T_s S$, is always unique.

Fact: universal deformation may not exist.

Thm (Kuranishi):

Every compact complex manifold X_0 admits a versal deformation $\begin{matrix} X \\ \downarrow \\ S \end{matrix}$, $\underbrace{T_s S \xrightarrow{\sim} H^1(X_0, T_{X_0})}_{\text{Kodaira - Spencer map}}$.

S may not be smooth.

If X_0 admits a versal deformation $\begin{matrix} X \\ \downarrow \\ S \end{matrix}$ with S smooth, we call the deformation of X_0 is unobstructed.

(Tian - Todorov): Calabi - Yau $\Rightarrow$ unobstructed.