Lecture 15 27-April 2022

Remark $f \in D \Rightarrow \exists\, u \in HD, \; v \in D_0$ s.t.

$$f = u + v$$

f superharmonic $\Rightarrow$ $v = f - u$ superharmonic and $v \in D_0$

$\Rightarrow v \geq 0$.

Exercise. Let $u, \hat{u}$ harmonic.

$$v^+(x) := \max\{u(x), \hat{u}(x)\}$$

$$v^-(x) := \min\{u(x), \hat{u}(x)\}$$

$\Rightarrow$ v^+ is subharmonic, i.e. $v^+(x) \leq (Pv^+)(x)$

v^- is superharmonic, $v^-(x) \geq (Pv^-)(x)$.

Thm. Suppose $u \in HD$. Then $\exists\ u_1, u_2 \in HD$ non-negative s.t.

$$u = u_1 - u_2$$

Pf:

$$\left.\begin{aligned} f_1 &:= \max\{u, 0\} \geq 0 \\ f_2 &:= \max\{-u, 0\} \geq 0. \end{aligned}\right\} f_1, f_2 : V \to \mathbb{R}$$

$\Rightarrow$ f_1, f_2 subharmonic

$\Rightarrow$ $-f_1$, $-f_2$ superharmonic

Royden's decomposition $\Rightarrow$ $\exists\ u_1, u_2 \in HD$, $v_1, v_2 \in D_0$ s.t.

$-f_1 = u_1 + v_1 \Rightarrow v_1 \geq 0$ since $-f_1$ superharmonic

$-f_2 = u_2 + v_2 \Rightarrow v_2 \geq 0.$

$$-u_1 = f_1 + v_1 \geq 0$$

$$-u_2 = f_2 + v_2 \geq 0$$

$$\Rightarrow \quad \underset{\swarrow HD}{u} = f_1 - f_2 = \underset{\swarrow HD}{(-u_1+u_2)} + \underset{\swarrow D_0}{(-v_1+v_2)}$$

$$= u + 0.$$

By uniqueness of Royden's decomposition $\Rightarrow$

$$u = -u_1 + u_2$$

$$0 = -v_1 + v_2$$

$$\Rightarrow \quad u = (-u_1) - (-u_2)$$

Thm For every $h \in HD$, $\exists$ a sequence h_n of bounded functions in HD s.t.

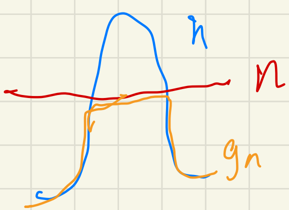
$$\|h - h_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

If $h \geq 0$, then $0 \leq h_n \leq h$.

Pf. Only consider (Γ, r) transient.

Assume $h \geq 0$.

① Construct sequence of bounded functions.



$$g_n(x) := \min\{h(x), n\} \quad \forall x \in V.$$

$$g_n(x) \leq n \quad \forall x$$

$\Rightarrow$ g_n is bounded.

② Royden' decomposition $\Rightarrow$ $h_n \in HD$, $s_n \in D_0$ s.t.

$$g_n = h_n + s_n$$

Note: g_n superharmonic $\Rightarrow$ s_n superharmonic and $s_n \in D_0$

$$\Rightarrow \quad s_n \geq 0$$

$$\Rightarrow \quad h_n \leq g_n \overset{h}{\leq} n$$

$\Rightarrow$ $\{h_n\}$ bounded harmonic function

③ Show: $\{h_n\}$ converges to h.

$$D(h-g_n) = D(h-h_n-s_n)$$

$$= D(h-h_n) + D(s_n) - 2\left[\overset{HD}{\overset{\Downarrow}{h-h_n}}, \overset{D_0}{\overset{\Downarrow}{s_n}}\right] \to 0.$$

$$D(h-h_n) + D(s_n) = D(h-g_n) \xrightarrow[\text{(Exercise)}]{} 0 \quad \text{as } n\to\infty.$$

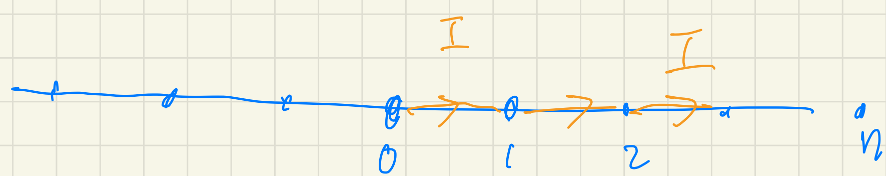
$$\Rightarrow \begin{cases} D(h-h_n) \to 0 \\ D(s_n) \to 0 \end{cases} \quad \text{as } n \to \infty. \quad \text{since } s_n \to 0.$$

$$\|h-h_n\|^2 = \underbrace{(h(0)-h_n(0))^2}_{\to 0} + \underbrace{D(h-h_n)}_{\to 0}$$

$$\Rightarrow \lim_{n\to\infty} \|h-h_n\| = 0$$

Ex: $(\mathbb{Z}, r)$ $\quad r_{n,n+1} = \cancel{\frac{1}{n^2}}\ \frac{1}{n}$ $\quad W(I) = \sum I^2 \cdot r$

$V = Ir$



$$\Rightarrow \quad W(I) = \sum \frac{1}{n^2} < \infty.$$

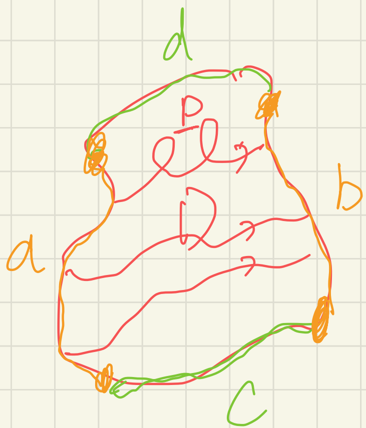
$$u(n) - u(0) = \sum \frac{1}{n^2} < \infty.$$

$\Rightarrow$ not a good example.

Extremal length.

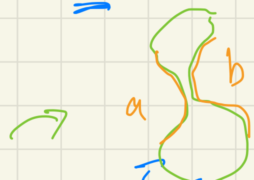
Background from classical theory.

Let D open set in $\mathbb{C}$, topologically quadrilateral.



$\underline{P}$ set of all paths from a-side to b-side.

rectifiable. Let $\sigma: D \to \mathbb{R}_{>0}$.



For each path $p \in \underline{P}$

$$L_\sigma(p) := \int_p \sigma \, |dz|$$

$$L_\sigma(\underline{P}) = \inf_{p \in \underline{P}} L(p)$$

$\Rightarrow \partial D = a + c + b + d$

D has four corners.

$$\mathrm{Area}_\sigma(D) = \iint_D \sigma^2 \, dx\, dy,$$

Extremal length: $\lambda(\underline{P}) := \sup_{\sigma : D \to \mathbb{R}_{>0}} \dfrac{L_\sigma(\underline{P})^2}{\mathrm{Area}_\sigma(D)}$

Rmk: $\lambda(\underline{P})$ is conformal invariant.

i.e. if $f : D \to \mathbb{C}$ conformal injective,

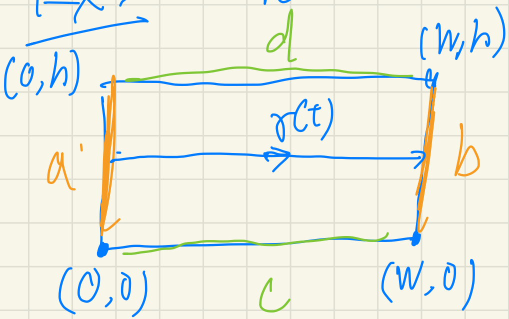
$\Rightarrow$ $\hat{D} := f(D)$,

$\hat{\underline{P}} := \{ f(p) \mid p \in \underline{P} \}$.

$\Rightarrow$ $\lambda(\hat{\underline{P}}) = \lambda(\underline{P})$.

Rmk: $\inf_{\sigma} \frac{L_{\sigma}^{2}(\underline{P})}{\mathrm{Area}_{\sigma}(D)} = 0$. not interesting.

Ex. D is rectangle



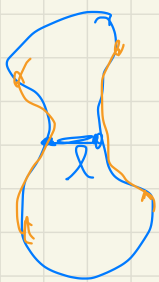
Claim: $\lambda(\underline{P}) = \frac{w}{h}$

Check: Consider $\sigma \equiv 1$.

$$L_{\sigma}(\underline{P}) = w$$

$$\mathrm{Area}_{\sigma}(D) = w \cdot h$$

$$\lambda(\underline{P}) = \sup_{\sigma} \frac{L_{\sigma}^{2}(\underline{P})}{\mathrm{Area}_{\sigma}(D)} \geq \frac{w^2}{wh} = \frac{w}{h}$$



Given any $\sigma : D \to \mathbb{R}_{>0}$, $\quad \ell := L_\sigma(\mathcal{P}) > 0$.

$$\gamma(t) = wt + iy \qquad \text{for } t \in [0,1],$$

$$\Rightarrow \quad \gamma \in \mathcal{P}.$$

$$\ell \le L_\sigma(\gamma) = \int_0^1 \sigma \left|\frac{d\gamma}{dt}\right| dt$$

$$= \int_0^1 \sigma\, w\, dt \qquad x = wt.$$

$$\ell \cdot h \le \int_0^h \int_0^1 \sigma\, w\, dt\, dy$$

$$= \int_0^h \int_0^w \sigma\, dx\, dy$$

Cauchy-Schwarz $\leq \sqrt{\left(\int_0^h\int_0^w \sigma^2\,dx\,dy\right)\cdot\left(\int_0^h\int_0^w dx\,dy\right)}$

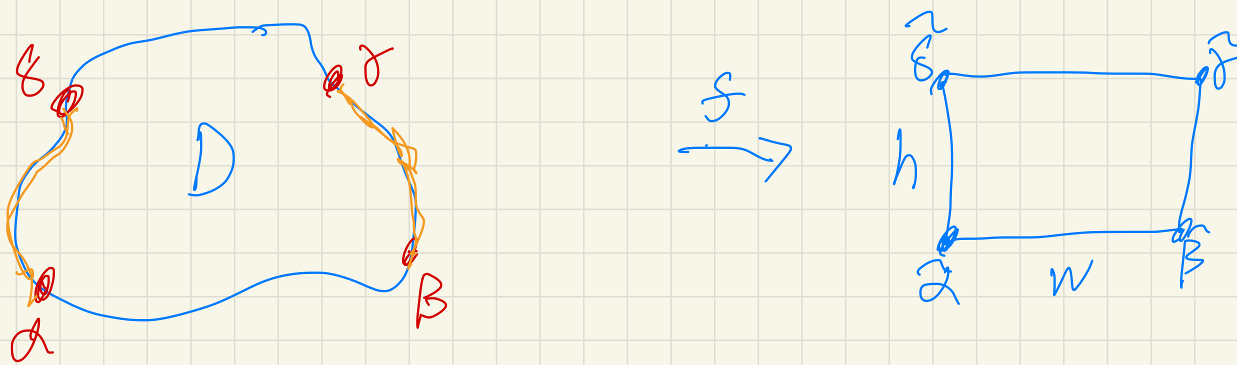
$$\Rightarrow \quad \ell^2 h^2 \leq \mathrm{Area}_\sigma(D)\cdot w\cdot h$$

$$\Rightarrow \quad \lambda(P) = \sup_\sigma \frac{\ell^2}{\mathrm{Area}_\sigma(D)} \leq \frac{w}{h} \quad \text{holds for } \sigma.$$

Conclusion: $\lambda(P) = \dfrac{w}{h}$.

Rmk: $(\lambda(P))^{-1} = \inf\limits_{\substack{\sigma \text{ s.t.}\\ L_\sigma(P)\geq 1}} \mathrm{Area}_\sigma(D)$

Thm

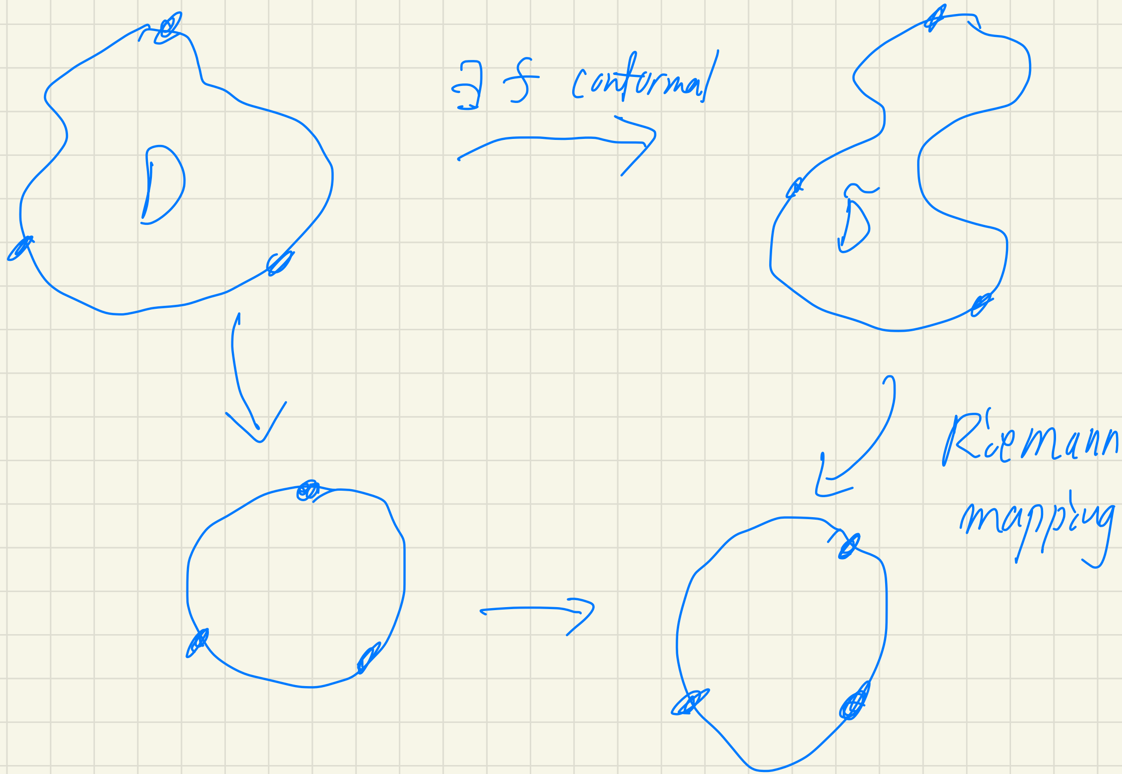


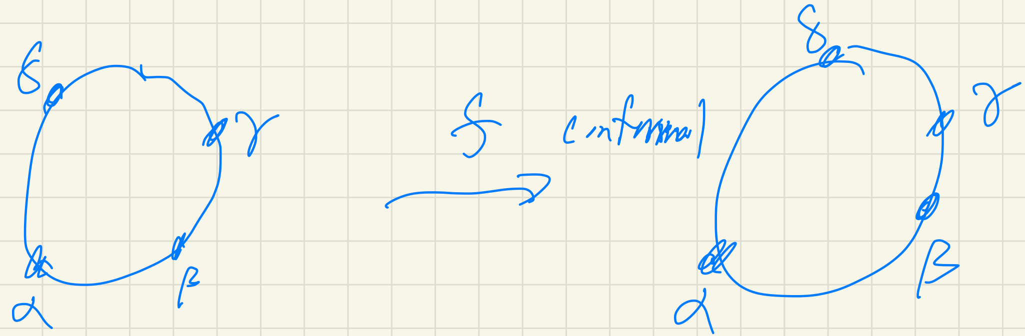
$\exists\ f$ conformal s.t. $f(\alpha) = \tilde{\alpha}$, $f(\beta) = \tilde{\beta}$ .....

$$\Longleftrightarrow \quad \lambda(P) = \frac{w}{h}$$

conformal modulus of D with 4-mark points.

Remark: Regions with 3-points are conformal equivalent





f exists mapping $f(\alpha)=\tilde{\alpha}, \ldots, f(\delta)=\tilde{\delta}$

$\iff$ cross ratio of $\{\alpha, \beta, \gamma, \delta\}$

$\quad\quad\quad\quad ||$

cross ratio of $\{\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}\}$

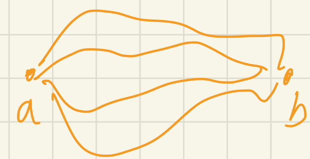
$$cr(\alpha, \beta, \gamma, \delta) = \frac{\alpha-\beta}{\beta-\gamma} \, \frac{\gamma-\delta}{\delta-\alpha}$$

Def (Γ, r) network.

$\underline{P}$ family of (finite or infinite) paths in Γ.

$p \in \underline{P}$ path $\Rightarrow$ $E(p)$ collection of edges in p.

Extremal length $\lambda(\underline{P})$



$$\left[\lambda(\underline{P})\right]^{-1} = \inf\{ W(I) \mid I \text{ 1-chain s.t. } I \in \mathcal{Q}(\underline{P}) \}$$

where $\mathcal{Q}(\underline{P})$ is collection of 1-chains I s.t. $\forall p \in \underline{P}$.

$$\sum (\Delta V)_{xy} = \sum_{xy \in E(p)} r(x,y)\,|I(x,y)| \geq 1$$

Idea: Energy $= V I$