set $Y \xrightarrow{g} T$, $\varphi: \mathrm{Cor}_S \to \mathrm{Cor}_T$

$(\varphi^*, \varphi_\#)$ $\varphi^*: Sh(S) \to Sh(T)$ φ^* takes acyclic projective complexes to acyclic projective complexes

Prop 5.12 We have an exact functor

$$L\varphi^*: D^-(S) \to D^-(T)$$
$$K \mapsto \varphi^* P$$

where $P \to K$ is a projective resolution.

Proof. By Prop 5.11, the class of projective complexes is adapted (see [GM03, III 6.3]) to the functor φ^*. We apply [GM03, III 6.?] to conclude. (Gelfand, Manin)

In the sequel, we write φ^* in place of $L\varphi^*$ for convenience.

Prop 5.13. 1) The category $D^-(S)$ is endowed with a tensor product defined by

$$\otimes_S: D^-(S) \times D^-(S) \to D^-(S)$$
$$(K, L) \mapsto P \otimes Q$$

where P, Q are projective resolutions of K, L, respectively and $P \otimes Q = \mathrm{Tot}(\{P_i \otimes Q_j\})$. Moreover, for any $K \in D^-(S)$, the functor $K \otimes_S -$ is exact.

2) Suppose that $f: S \to T$ is smooth. Then there is an exact functor

$$f_\#: D^-(S) \to D^-(T)$$
$$K \mapsto f_\# P$$

where $P \to K$ is a projective resolution.

3) Suppose that $f: S \to T$ is in Sm/k. Then there is an exact functor

$$f^*: D^-(T) \to D^-(S)$$
$$K \mapsto f^* P$$

where $P \to K$ is a projective resolution.

Proof: 1) Let $Y \in Sm/S$. We take φ to be $S \leftarrow Y \to S$, so $\varphi^* F = \ldots$

Given an acyclic projective complex P and a projective sheaf F, the complex $F \otimes P$ is also acyclic by Prop 5.11. So for any projective complex K, the complex $P \otimes K$ is also acyclic by spectral sequence of the double complex $\{P_i \otimes K_j\}$. Then for any projective complexes P, Q, R with a quasi-isomorphism $a: P \to Q$, we have

$$C(a \otimes R) = C(a) \otimes R$$

and the latter is acyclic. Here $(C(a))^i = P^{i+1} \oplus Q^i$ is the mapping cone of a.

So $a \otimes R$ is also a quasi-isomorphism.

2) Take $(Y, S, T) = (S, S, T)$ and apply Prop 5.12.

3) Take $(Y, S, T) = (T, S, T)$ . . . □

Prop 5.14. Let $f: S \to T$ be a smooth morphism in Sm/k. We have an adjoint pair

$$f_\#: D^-(S) \rightleftarrows D^-(T): f^*$$

Proof. We have an adjunction

$$f_\#: K^-(S) \rightleftarrows K^-(T): f^*$$

simply by the adjunction between sheaves. Since f^* has both left and right adjoints, it is exact so $Lf^* = f^*$. Suppose that $K \in D^-(S)$, $L \in D^-(T)$ and that $P \to K$ is a projective resolution.

We construct a morphism

$$\theta: \mathrm{Hom}_{D^-(S)}(f_\# K, L) \to \mathrm{Hom}_{D^-(T)}(K, f^* L)$$

as follows. Suppose that $s \in \mathrm{Hom}_{D^-(S)}(f_\# K, L)$ is written as a right roof

$$s = \quad f_\# P \xrightarrow{a} R \xleftarrow{b} L.$$

We define $\theta(s)$ to be the map

$$K \Leftarrow P \xrightarrow{a'} f^* R \xleftarrow{f^* b} L.$$

The $f^* b$ is a quasi-iso since f is exact.

Next, we construct a morphism

$$\xi: \mathrm{Hom}_{D^-(T)}(K, f^* L) \to \mathrm{Hom}_{D^-(S)}(f_\# K, L).$$

Suppose that $t \in \mathrm{Hom}_{D^-(T)}(K, f^* L)$ and $t \circ p$ is written as a left roof

$$P \xLeftarrow{a} R \xrightarrow{b} f^* L$$

where R could be taken projective. Define $\xi(t)$ as

$$f_\# P \xLeftarrow{f_\# a} f_\# R \xrightarrow{b'} L$$

by Prop 5.13. One checks that θ, ξ are inverse to each other □

For theory on $D(S)$, refer to Déglise's and Cisinski paper "Stable and local homological algebra . . .".

Now let us get the homotopy relation $X \times \mathbb{A}^1 \sim X$ involved.

Def 5.15. An $F \in PSh(S)$ is called homotopy invariant if for every $X \in Sm/S$, the map $F(X) \xrightarrow{p^*} F(X \times \mathbb{A}^1)$ $(p: X \times \mathbb{A}^1 \to X)$ is an isomorphism.

Since p has a section, p^* is split injective. So F is h.i. $\iff$ p^* is surjective.

Lem 5.16 F is homotopy invariant iff $\quad X \overset{i_1}{\underset{i_0}{\rightrightarrows}} X \times \mathbb{A}^1$

$$i_0^* = i_1^*: F(X \times \mathbb{A}^1) \to F(X) \quad \text{for all } X.$$

Proof: We only have to show the sufficiency. Denote by $m: \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$, $(x, y) \mapsto xy$ the multiplication map. We have a commutative diagram

$$\begin{tikzcd}
 & F(X \times \mathbb{A}^1) \arrow[r, "i_0^*"] \arrow[dl, equal] \arrow[d, "(id_X \times m)^*"] & F(X) \arrow[d, "p^*"] \\
F(X \times \mathbb{A}^1) & F(X \times \mathbb{A}^1 \times \mathbb{A}^1) \arrow[l, "(id_X, 1, id_{\mathbb{A}^1})^*"'] \arrow[r, "(id_X, 0, id_{\mathbb{A}^1})^*"'] & F(X \times \mathbb{A}^1)
\end{tikzcd}$$

But $(id_X, 1, id_{\mathbb{A}^1})^* = (id_X, 0, id_{\mathbb{A}^1})^*$, so $p^* i_0^* = id$. □

Lem 5.17 For any $F \in PSh(S)$, the maps $i_0^*, i_1^*: (C_* F)(X \times \mathbb{A}^1) \to (C_* F)(X)$ are chain homotopic.

$(C_* F)_n = F^{\Delta^n}$, $\Delta^n = \mathrm{Spec}\, k[x_0, \cdots x_n]/(\sum x_i - 1) \cong \mathbb{A}^n$

Proof: For any $i = 0, \cdots, n$, define $\theta_i: \Delta^{n+1} \to \Delta^n \times \mathbb{A}^1$

$$\underset{j\text{-th}}{v_j = (0, \cdots, 1, \cdots 0)} \mapsto \begin{cases} (v_j, 0) & 0 \le j \le i \\ (v_{j-1}, 1) & i < j \le n+1 \end{cases}$$

Each θ_i induces a map $h_i = F(id_X \times \theta_i): F^{\Delta^n}(X \times \mathbb{A}^1) \to F^{\Delta^{n+1}}(X)$. Then the same

$$F^{\Delta^n}(X \times \mathbb{A}^1) = F(X \times \mathbb{A}^1 \times \Delta^n), \quad F^{\Delta^{n+1}}(X) = F(X \times \Delta^{n+1})$$

proof like Prism decomposition as in topology that $s_n = \sum_i (-1)^i h_i$ is a chain homotopy from i_1^* to i_0^*. □

Prop 5.18 For any $F \in PSh(S)$, the homology presheaves

$$H_n(C_* F): X \mapsto H_n((C_* F)(X))$$

are homotopy invariant.

$\boxed{\Delta^* \text{ is homotopy invariant}}$

Proof: By Lem 5.17, i_0^*, i_1^* of $H_n(C_* F)$ are equal. So we conclude by Lem 5.16. □

Def 5.19 A full additive subcategory $\mathcal{D}$ of a triangulated category $\mathcal{C}$ is called thick if

1) Let $A \to B \to C \to A[1]$ be a distinguished triangle. Then if two out of A, B, C are in $\mathcal{D}$ so is the third.

2). If $A \oplus B \in \mathcal{D}$, then $A, B \in \mathcal{D}$.

Def Let $\mathcal{C}$ be a category and $S \subseteq \mathcal{C}$ be a class of maps. We say that S is left localizing if: 1) $id_X \in S$, $\forall X \in \mathcal{C}$, $f, g \in S \Rightarrow g \circ f \in S$

(right $\iff$ left in $\mathcal{C}^\circ$)

2) Any $X \to Y$ with $X \Rightarrow X'$ in S can be completed to

$$\begin{tikzcd} X \arrow[r] \arrow[d, Rightarrow, "\in S"] & Y \arrow[d, Rightarrow, "\in S"] \\ X' \arrow[r] & Y' \end{tikzcd}$$

3). $X' \overset{\delta}{\Rightarrow} X \overset{\alpha}{\underset{\beta}{\rightrightarrows}} Y$, $\alpha \circ \delta = \beta \circ \delta$, then $\exists\, Y \overset{\gamma}{\Rightarrow} Y'$ s.t. $\gamma \circ \alpha = \gamma \circ \beta$.

Prop 5.20 Let $\mathcal{D}$ be a thick subcategory of $\mathcal{C}$ (tri.cat.). Define $W_{\mathcal{D}}$ to be those maps whose mapping cone are in $\mathcal{D}$. Then $W_{\mathcal{D}}$ is a left and right localizing system. Consider the category $\mathcal{C}[W_{\mathcal{D}}^{-1}]$ with objects being those of $\mathcal{C}$, morphisms being left or right roofs ($X \overset{s}{\Leftarrow} Z \to Y$). Then $\mathcal{C}[W_{\mathcal{D}}^{-1}]$ is a triangulated category. Its distinguished triangles are those isomorphic to distinguished triangles coming from $\mathcal{C}$. Moreover, if $\mathcal{E}$ is another triangulated cat and $F: \mathcal{C} \to \mathcal{E}$ is an exact functor with $F(\alpha) = 0$ $\forall \alpha \in \mathcal{D}$, then $\exists!\, \mathcal{C}[W_{\mathcal{D}}^{-1}] \to \mathcal{E}$ (exact) s.t.

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, "F"', "{F(\alpha)=0,\ \alpha \in \mathcal{D}}"] & \mathcal{C}[W_{\mathcal{D}}^{-1}] \arrow[dl, "\exists!"] \\ \mathcal{E} & \end{tikzcd}$$

Proof: See [GM03, IV.2 Ex 4]. □

Def 5.21 Define $\mathcal{C}_{\mathbb{A}}$ to be the smallest thick subcategory in $D^-(S)$ s.t.

1) The cone of $\mathbb{Z}(X \times \mathbb{A}^1) \to \mathbb{Z}(X)$ is in $\mathcal{C}_{\mathbb{A}}$ for every $X \in Sm/S$.

2). $\mathcal{C}_{\mathbb{A}}$ is closed under any direct sum that exists in $D^-(S)$.

We say that $f \in D^-(S)$ is an $\mathbb{A}^1$-weak equivalence if $f \in W_{\mathcal{C}_{\mathbb{A}}}$. Define

$$DM^{eff,-}(S) = D^-(S)[W_{\mathcal{C}_{\mathbb{A}}}^{-1}]$$

to be the category of effective motives over S.

Lem 5.22. The smallest class in $D^-(S)$ which contain all the $\mathbb{Z}(X)$ and is closed under quasi-isomorphisms, direct sums, shifts and cones is all of $D^-(S)$.

Proof: First we show that for any complex $D_\cdot$, if all D_n are in the class, then so is $D_\cdot$. If $\beta_n D_\cdot$ is the truncation $0 \to D_n \to D_{n-1} \to \cdots$ of $D_\cdot$, then $D_\cdot = \varinjlim_n \beta_n D_\cdot$. We have a distinguished triangle

$$D_n \to \beta_{n-1} D \to \beta_n D \to D_n[1]$$

$$\beta_{n-1} D = (0 \to D_{n-1} \to D_{n-2} \cdots)$$

So each $\beta_n D$ belongs to the class. Moreover, since there is an exact sequence

$$0 \to \bigoplus \beta_n D \to \bigoplus \beta_n D \to D_\cdot \to 0$$

it follows that $D_\cdot$ is in the class.

Finally, for each sheaf F, there is a free resolution $L_\cdot \to F$, we conclude □

$$\to \bigoplus \mathbb{Z}(X) \to K \to \bigoplus_{\alpha \in F(X)} \mathbb{Z}(X) \xrightarrow{i_\alpha} F \to 0$$