Lecture 12 (Next lecture this Sunday)

For recurrent network, replace $G_n(x,y;b_n) \to G_n(x,y;q)$

(Recall: $b_n \notin V_n$) ($q \in V_n$ for all n)

Thm (Γ, r) recurrent network, Fix $q \in V$. For any 0-chain τ, consider $f(x) := -\frac{\tau(x)}{c(x)}$. Suppose

$\tau(q)$ does affect this summation

$$\left(\sum_{x,y \in V} c(x)\, G(x,y;q)\, |f(x) - f(y)| \right) < \infty$$

Recall: $G(x,y;q) = 0$ if $x = q$ or $y = q$

Then $\exists$ $I \in H_1$ s.t.

$$\partial I(x) + \tau(x) = 0 \qquad \forall x \in V - \{q\},$$

$$\langle I, z \rangle = 0 \qquad \forall \text{ finite cycles.}$$

① $G(|f|;q)(x) = \sum_{y\in V} G(x,y;q)\,|f(y)| < \infty \quad \forall x.$

② $u(x) := \sum_{y\in V} G(x,y;q)\, f(y)$ is potential of I.

③ $W(I) = \sum_{x,y\in V} c(x)\, G(x,y;q)\, |f(x)-f(y)|$.

$$\left(= \sum_{x,y\in V} c(x,y)\, |u(x)-u(y)|^2 \right)$$

Conclusion: $z \in \partial H_1 \iff \{W(I_n)\}$ is bounded

If exists, then it is given by $G(f)$,

and $f = -\frac{z}{c}$

$G(|f|)(x) < \infty \quad \forall x \Rightarrow \quad G(f)(x)$ is well-defined

$\Leftarrow$

$f(x) = -\frac{\nu(x)}{c(x)}$ $\qquad u = G(f)$ is potential.

Ex: (Γ, r) recurrent. $\exists$ superharmonic function non constant?

III. Section 7. Dirichlet's principles and Rayleigh's principles.

<u>Thm</u> (Γ, r) infinite network. Let $a \in V$.

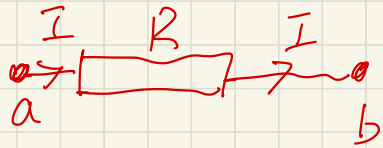
$$\left(R_{eff}(a,\infty)\right)^{-1} = \inf\{D(u) \mid u(a)=1,\ u \in \ell_0\}$$

$\updownarrow$

u has finite support.

Recall: ① Energy $= I^2 R = \frac{V^2}{R} = IV$ — $(*)$

Know: $V = IR$



$(*) \Rightarrow$ Energy $= \begin{cases} R & \text{if } I \text{ is unit current} \\ \frac{1}{R} & \text{if difference of voltage is } 1 = V \end{cases}$

② $R(a,\infty) = \frac{G(a,a)}{c(a)} = \begin{cases} \infty & \text{if recurrent} \\ < \infty & \text{if transient.} \end{cases}$

Corollary (Γ, r) recurrent $\Longleftrightarrow$ For some vertex $a \in V$, there exists $\{u_n : V \to \mathbb{R}\}_{n \in \mathbb{N}}$ s.t. ① $u_n(a) = 1$, $u_n \in \ell_0$

② $\lim_{n \to \infty} D(u_n) = 0$

Thm Suppose (Γ, r) recurrent and $z \in \partial H_1$.

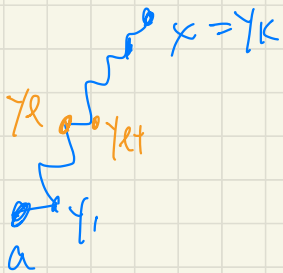
If $\sum_{x \in V} |z(x)| < \infty$, then $\sum_{x \in V} z(x) = 0$.

Pf: Assume $\{u_n : V \to \mathbb{R}\}_{n \in \mathbb{N}}$ s.t. $u_n(a) = 1$ and $u_n \in \ell_0$

and $\lim_{n \to \infty} D(u_n) = 0$.

For any $x \in V$, $\quad |u_n(x) - u_n(a)| < \left| \sum_{\ell=1}^{k} \left(u_n(y_k) - u_n(y_{k-1}) \right) \right|$

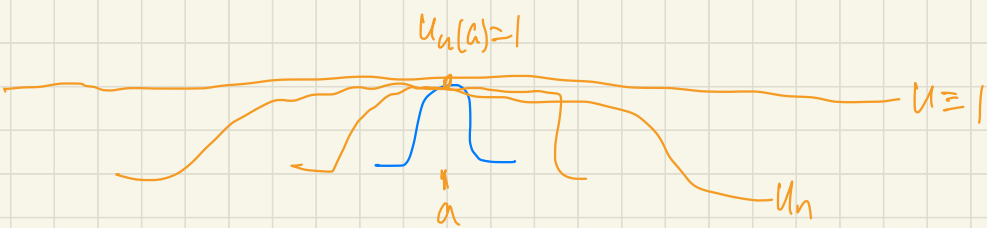
$$\| \qquad |u_n(x) - 1| \leq \sum 1 \cdot |u_n(y_k) - u_n(y_{k-1})|$$



$$\leq \sqrt{\sum_{\text{const}} \sum_{\text{const}} \left(\sum_{\text{const}} |u_n(y_k) - u_n(y_{k-1})| \right)^2}$$

$$\leq \sqrt{D(u_n)} \cdot \sqrt{k} \longrightarrow 0$$

Claim: $\quad \lim_{n \to \infty} u_n(x) = 1 = u_n(a)$.



For any $x, y \in E$, $|u_n(x) - u_n(y)| < \sqrt{D(u_n)} \to 0$ as $n \to \infty$.

$I \in H_1$, $\partial I + z = 0$.

Consider for each n,

$$\sum_{x \in V} z(x) \xleftarrow[\text{as } n \to \infty]{} \sum_{x \in V} z(x)\, u_n(x) = \sum_{x \in V} (\partial I(x))\, u_n(x)$$

$$= \sum_{x,y \in Y} I(x,y)\,(u_n(x) - u_n(y))$$

$$\overset{\text{Cauchy Schwarz}}{\leq} \left(\sum_{x,y \in Y} r(x,y)\, I^2(x,y)\right)\left(\sum_{x,y \in Y} c(x,y)\,(u_n(x) - u_n(y))^2\right)$$

$$= W(I)\, D(u_n) \to 0 \quad \text{as } n \to \infty$$

$$\Rightarrow \quad \sum_{x \in V} \tau(x) = 0.$$

Q: If (Γ, r) transient, where does the argument fail?

Rayleigh's principle.

(1). Given (Γ, r_1), (Γ, r_2).

If $r_1(B) \leq r_2(B) \quad \forall B \in Y$.

$$R_1(a,b) \leq R_2(a,b) \quad \text{for any } a, b \in V.$$

a, b might not be adjacent

(2). $(\tilde{\Gamma}, \tilde{r}) \subset (\Gamma, r)$ is a sub-network with induced resistance.

$\parallel$

(Γ, r_2) where $r_2(B) = \begin{cases} r(B) & \text{if } B \in \tilde{\Gamma} \\ \infty & \text{if } B \notin \tilde{\Gamma} \end{cases}$

$\Rightarrow \quad r_2(B) \geq r(B)$

$\tilde{r}(B) = r(B) \quad \forall B \in \tilde{\Gamma}$.

Take $a, b \in \tilde{V}$.

$$\tilde{R}(a,b) \geq R(a,b).$$

(3). $(\tilde{\Gamma}, \tilde{r}) \subset (\Gamma, r)$ finite $\rightsquigarrow (\Gamma', r')$ by shorting all vertices outside $\tilde{\Gamma}$ to $\{b\}$.

$\parallel$

(Γ, r_2)

$r_2(B) = \begin{cases} r(B) & \text{if } B \cap \tilde{V} \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$

$\Rightarrow \quad r_2(B) \leq r(B)$.

Take $a, b \in \tilde{V}$.

$$R'(a,b) \leq R(a,b)$$

Def $\Gamma = (V, Y)$ infinite graph.

Partition of Γ is $\{V_n\}_{n\in\mathbb{N}} \subset V$ s.t.

(1) $\bigcup_{n=1}^{\infty} V_n = V$

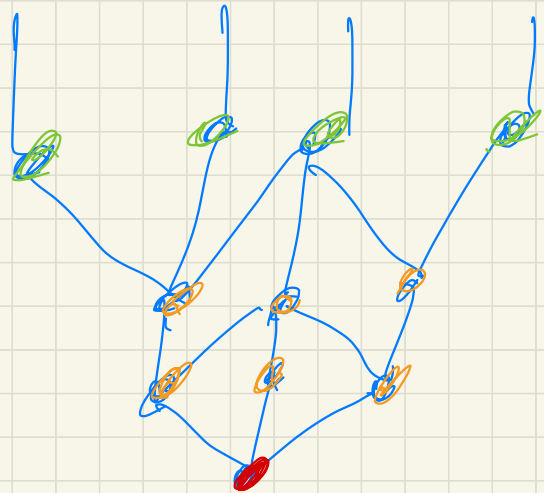
(2). $V_n \cap V_m = \phi$ if $n \neq m$.

(3). $x \in V_n$, $xy \in Y \Rightarrow y \in V_{n-1} \cup V_n \cup V_{n+1}$

$\Rightarrow$ $\{V_n\}$ a canonical decomposition.

Thm (*) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sum_{\substack{x\in V_{n-1}\\ y\in V_n}} c(x,y)} = \infty \Rightarrow (\Gamma, r)$ recurrent.

Pf:

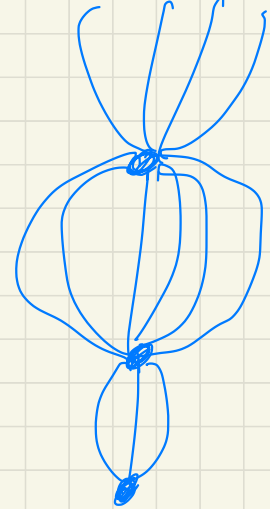


V_3

V_2

V_1

a



a'

$$R_{eff}(a,\infty) \geq R_{eff}(a',\infty) = \sum_{n=1}^{\infty} r_{n,n+1} = \sum_{n=1}^{\infty} \frac{1}{\sum_{\substack{x\in V_n \\ y\in V_{n+1}}} c_{xy}}$$

$$\frac{1}{r_{n,n+1}} = \sum_{\substack{x\in V_n \\ y\in V_{n+1}}} \frac{1}{r_{xy}} \Rightarrow r_{n,n+1} = \frac{1}{\sum_{\substack{x\in V_n \\ y\in V_{n+1}}} c_{xy}}$$

$(*) = \infty \Rightarrow R_{eff}(a,\infty) = \infty$

$\Rightarrow$ recurrent.

<u>Corollary</u> $(**)$ (Γ, r) and $r \equiv 1$. Fix $o \in V$.

$$\tilde{V}_n := \{ x \in V \mid d(x,o) \le n \}.$$

$$V_n := \tilde{V}_{n+1} - \tilde{V}_n \Rightarrow \{V_n\} \text{ canonical decomposition.}$$

Consider $e_n :=$ number of edges x,y s.t. $x,y \in \tilde{V}_n$

Suppose $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e_{n+1} - e_n} = \infty \Rightarrow (\Gamma, r)$ is ~~transient~~ recurrent.

In particular if $e_n \leq \text{const.}\, n^2 \log n$,

then (Γ, r) is current.

<u>Ex</u> $\Gamma = \mathbb{Z}$, $e_n \sim n \Rightarrow \mathbb{Z}$ recurrent.

$\Gamma = \mathbb{Z}^2$, $e_n \sim n^2 \Rightarrow \mathbb{Z}^2$ recurrent.

$\Gamma = \mathbb{Z}^k$, $k \geq 3$. are transient.

Next lecture is this Sunday, 15:20 – 16:50.

Idea: Corollary (2.10) is used to prove recurrence.

Existence of non-const $\begin{matrix}\text{non-negative superharmonic} \\ \text{finite energy superharmonic}\end{matrix}$ function $\Rightarrow$ transience.