Recall: $\varphi:\mathcal{C}\to\mathcal{D}$, $\varphi_*:\mathrm{Preshv}(\mathcal{D},\mathcal{M})\to\mathrm{Preshv}(\mathcal{C},\mathcal{M})$

$$F\longmapsto F\circ\varphi$$

φ_* has an left adjoint φ^*.

Def 5.2 Suppose that $f:S\to T$ is a morphism in Sm/k. We have a functor

$$\begin{aligned}\varphi^f: Cor_T &\to Cor_S\\ X&\longmapsto X\times_T S\\ f&\longmapsto f\times_T S.\end{aligned}$$

Then the Lem 5.1 provides us adjoint functors.

$$f^*: PSh(T)\rightleftarrows PSh(S): f_* = (\varphi^f)_*$$
$$f^*: Sh(T)\rightleftarrows Sh(S): f_*$$

In $Sh(S)$, $X\in Sm/S$, $\mathbb{Z}_S(X)(Y)=Cor_S(Y,X)$

Prop 5.3: Suppose that $f:S\to T$ is a morphism in Sm/k.

1) $f^*\mathbb{Z}_T(Y)=\mathbb{Z}_S(Y\times_T S)$ for any $Y\in Sm/T$

2). $(f_*F)^Y = f_*(F^{Y\times_T S})$ for any $F\in Sh(S)$ and $Y\in Sm/T$.

$F^X(Y)=F(X\times_S Y)$, $X,Y\in Sm/S$

3). $\underline{Hom}_T(F,f_*G)=f_*\underline{Hom}_S(f^*F,G)$ for any $F\in Sh(T)$ and $G\in Sh(S)$. $\underline{Hom}_S(F,G)(X)=Hom(F,G^X)$

4). $f^*F\otimes_S f^*G = f^*(F\otimes_T G)$ for any $F,G\in Sh(T)$. $(\otimes,\underline{Hom})$ adjunction

Proof: 1). We have $Hom_S(f^*\mathbb{Z}_T(Y),-)=Hom_T(\mathbb{Z}_T(Y),f_*-)=Hom_S(\mathbb{Z}_S(Y\times_T S),-)$.

2). $(f_*F)^Y(Z)=F((Y\times_T Z)\times_T S)\cong F((Z\times_T S)\times_S(Y\times_T S))\simeq (f_*(F^{Y\times_T S}))(Z)$, $Z\in Sm/T$.

3). For any $Y\in Sm/T$, we have

$$\begin{aligned}\underline{Hom}_T(F,f_*G)(Y)&=Hom_T(F,(f_*G)^Y)\\&=Hom_T(F,f_*(G^{Y\times_T S}))\\&=Hom_S(f^*F,G^{Y\times_T S})\\&=(f_*\underline{Hom}_S(f^*F,G))(Y).\end{aligned}$$

4). For any $H\in Sh(S)$, we have

$$\begin{aligned}Hom_S(f^*F\otimes_S f^*G,H)&=Hom_S(f^*G,\underline{Hom}_S(f^*F,H))\\&=Hom_T(G,f_*\underline{Hom}_S(f^*F,H))\\&=Hom_T(G,\underline{Hom}_T(F,f_*H))\\&=Hom_T(F\otimes_T G,f_*H)\\&=Hom_S(f^*(F\otimes_T G),H).\ \square\end{aligned}$$

Def 5.4 Suppose that $f:S\to T$ is a smooth morphism in Sm/k. So for every $X\in Sm/S$ is naturally an object in Sm/T. Moreover, there is a Cartesian diagram

$$\begin{tikzcd} X_1\times_S X_2 \arrow[r, "q_f"] \arrow[d] & X_1\times_T X_2 \arrow[d] \\ S \arrow[r, "\Delta"] & S\times_T S \end{tikzcd}$$

for any $X_1,X_2\in Sm/S$. So q_f is a closed immersion Thus we define

$$\begin{aligned}\varphi_f: Cor_S&\to Cor_T\\ X&\longmapsto X\\ g&\longmapsto q_{f*}(g).\end{aligned}$$

So by Lem 5.1, we obtain adjoint pairs:

$$f_\#: Sh(S)\rightleftarrows Sh(T):(\varphi_f)_*$$
$$f_\#: PSh(S)\rightleftarrows PSh(T):(\varphi_f)_*.$$

Prop 5.5. We have $(\varphi_f)_*=f^*$ for smooth $f:S\to T$.

Proof: For any $Y\in Sm/S$, the $id_Y\in Cor_T(Y,Y)=Cor_S(Y,Y\times_T S)$ is the initial element of C_Y: $\begin{cases}obj: X\to Y\times_T S, & X\in Sm/S,\ Y\in Sm/T.\\ mor: X\to Y_1\times_T S \to Y_2\times_T S & Y_1\to Y_2\end{cases}$ So we have $\downarrow f_*$

$$f^*(Y)=\varinjlim C_Y$$

$$(f^*F)(Y)=FY=((\varphi_f)_*F)(Y).\ \square$$

Prop 5.6: Let $f:S\to T$ be smooth.

1) $f_\#\mathbb{Z}_S(X)=\mathbb{Z}_T(X)$ for any $X\in Sm/S$.

2) $f^*(F^Y)=(f^*F)^{Y\times_T S}$ for any $F\in Sh(T)$ and $Y\in Sm/T$.

3). $\underline{Hom}_T(f_\# F,G)=f_*\underline{Hom}_S(F,f^*G)$ for any $F\in Sh(S)$ and $G\in Sh(T)$.

4). $f_\#(F\otimes_S f^*G)=(f_\# F)\otimes_T G$ for any $F\in Sh(S)$ and $G\in Sh(T)$

Proof: 1) For any $F\in Sh(T)$, we have

$$Hom_T(f_\#\mathbb{Z}_S(X),F)=Hom_S(\mathbb{Z}_S(X),f^*F)=(f^*F)(X)=FX.$$

2). For any $X\in Sm/S$, we have

$$(f^*(F^Y))(X)=F(Y\times_T X)=F((Y\times_T S)\times_S X)=(f^*F)^{Y\times_T S}(X).$$

3). For any $Y\in Sm/T$, we have

$$\begin{aligned}\underline{Hom}_T(f_\# F,G)(Y)&=Hom_T(f_\# F,G^Y)\\&=Hom_S(F,f^*(G^Y))\\&=Hom_S(F,(f^*G)^{Y\times_T S})\\&=\underline{Hom}_S(F,f^*G)(Y\times_T S)\\&=(f_*\underline{Hom}_S(F,f^*G))(Y).\end{aligned}$$

4). For any $H\in Sh(T)$, we have

$$\begin{aligned}Hom_T(f_\#(F\otimes_S f^*G),H)&=Hom_S(F\otimes_S f^*G,f^*H)\\&=Hom_S(f^*G,\underline{Hom}_S(F,f^*H))\\&=Hom_T(G,f_*\underline{Hom}_S(F,f^*H))\\&=Hom_T(G,\underline{Hom}_T(f_\# F,H))\\&=Hom_T((f_\# F)\otimes_T G,H)\ \square\end{aligned}$$

$K^-(S)[qis^{-1}]$ $\quad C^-(S)/$chain homotopy

Def 5.7. We define $D^-(S)$ (resp. $K^-(S)$) to be the derived (resp. homotopy) category of bounded above complexes of $Sh(S)$.

$F_n\to F_{n+1}\cdots\to 0\to 0\to\cdots$

We want to define $\otimes_S$, $f_\#$, f^* on $D^-(S)$.

Def 5.8. We call $F\in PSh(S)$ free if it is a direct sum of $\mathbb{Z}_S(X)$. We call F projective if it is a direct summand of a free F. We call $F\in Sh(S)$ free (resp. projective) if it is a sheafification of a free (resp. projective) presheaf. A bounded above complex of sheaves with transfers is call free (resp. projective) if all its terms are free (resp. projective).

Def 5.9 A projective resolution of $K\in C^-(S)$ is a quasi-isomorphism $P\to K$ with P being projective.

Now let $S,T\in Sm/k$ and Y be a scheme with morphism $\boxed{S\xleftarrow{f}Y\xrightarrow{g}T}$ where g is smooth. We consider functors:

$$\begin{aligned}\varphi: Cor_S\xrightarrow{\varphi^f} Cor_Y\xrightarrow{\varphi_g} Cor_T \qquad & \psi: Sm/S\to Sm/T\\ X\longmapsto X\times_S Y \qquad & X\longmapsto X\times_S Y\end{aligned}$$

determined by the triple (Y,S,T). (φ^*,φ_*)

Before stating the next result, recall that the $PSh(S)$ has enough projective objects. So it is possible to derive any left exact functors. (Say, to $\mathbf{Ab}$)

Prop 5.10 For any $F\in PSh(S)$, we have

$$((L_i\varphi^*)F^+)^+=((L_i\varphi^*)F)^+$$

for $i\geq 0$, where $L_i\varphi^*$ is the i-th left derived functor of φ^*.

Proof: We show at first that for any $F\in PSh(S)$ with $F^+=0$ we have

$$((L_i\varphi^*)F)^+=0,\quad i\geq 0\quad (*).$$

Suppose that this is proved. Then for any F, we consider the natural morphism

$$\theta: F\to F^+,$$

satisfying $(\mathrm{coker}\,\theta)^+=0=(\ker\theta)^+$. Hence for any $i\geq 0$, we have

$$((L_i\varphi^*)F^+)^+=((L_i\varphi^*)\,\mathrm{Im}\,\theta)^+=((L_i\varphi^*)F)^+$$

by long exact sequences. So the statement follows

Now we prove $(*)$ by induction on i. The case when $i=0$ is trivial, so suppose that it is true for $i<n$. For any $F\in PSh(S)$, we have a surjection

$$\bigoplus_{\alpha\in F(X)}\mathbb{Z}_S(X)\xrightarrow{i_\alpha}F$$

where i_α is given by the section α. Since $F^+=0$, there exists for any $\alpha\in F(X)$, $X\in Sm/S$, a Nisnevich covering $U_\alpha\to X$ s.t $\alpha|_{U_\alpha}=0$. Then the composite

$$\bigoplus_{\alpha\in F(X)}\mathbb{Z}_S(U_\alpha)\to\bigoplus_{\alpha\in F(X)}\mathbb{Z}_S(X)\xrightarrow{i_\alpha}F$$

$\check{C}(U_\alpha/X):\ \cdots\to\mathbb{Z}_S(U_\alpha\times_X U_\alpha)\to\mathbb{Z}(U_\alpha)\to\mathbb{Z}(X)\to\cdots$

is zero, so we obtain an exact sequence

$$0\to K\to\bigoplus_{\alpha\in F(X)}H_0(\check{C}(U_\alpha/X))\to F\to 0.$$

The Thm 2.27 implies that

$$H_p(\check{C}(U_\alpha/X))^+=0$$

for all α and $p\in\mathbb{Z}$, so $K^+=0$ as well. We have a hypercohomology spectral sequence (See Lem 1.20)

$$(L_p\varphi^*)H_q(\check{C}(U_\alpha/X))\Rightarrow(\mathbb{L}_{p+q}\varphi^*)\check{C}(U_\alpha/X).$$

So

$$((\mathbb{L}_n\varphi^*)\check{C}(U_\alpha/X))^+=((L_n\varphi^*)H_0(\check{C}(U_\alpha/X)))^+$$

by induction hypothesis. Since $\check{C}(U_\alpha/X)$ is a projective complex, we also have

$$((\mathbb{L}_n\varphi^*)\check{C}(U_\alpha/X))^+=(H_n(\varphi^*\check{C}(U_\alpha/X)))^+=(H_n(\check{C}(\psi(U_\alpha)/\psi(X))))^+ = 0.$$

Summarize, we have $((L_n\varphi^*)H_0(\check{C}(U_\alpha/X)))^+=0$. Hence

$$((L_n\varphi^*)F)^+=((L_{n-1}\varphi^*)K)^+=0$$

by long exact sequence and induction hypothesis. $\square$

Prop 5.11 The functor φ^* takes acyclic projective complexes of sheaves to acyclic projective complexes of sheaves.

Proof: For any projective $F\in Sh(S)$, $F=G^+$ for some projective $G\in PSh(S)$. So

$$((L_i\varphi^*)F)^+=((L_i\varphi^*)G)^+=0$$

for any $i>0$ by Prop 5.10. Let

$$0\to K\to F\to P\to 0$$

be a s.e.s. in $Sh(S)$ with $((L_i\varphi^*)P)^+=0$ $\forall i>0$. Then the sequence is still exact after applying φ^*. Moreover, if F is projective, we have $((L_i\varphi^*)K)^+=0$ for every $i>0$. This conclude the proof. $\square$