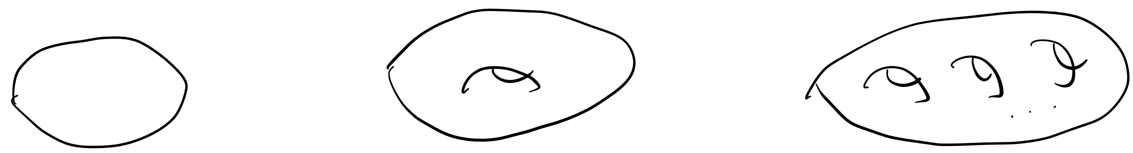
Riemann - Roch, genus formula, adjunction formula,

Serre duality.

Riemann surface : complex manifold of dimension 1

Compact.



$g=0$, S^1 $\qquad$ $g=1$ $\qquad$ $g \geq 2$

C: compact Riemann surface, $D = \sum_{i=1}^{k} m_i p_i$ formal sum of points

on C, $m_i \in \mathbb{Z} - \{0\}$,

$\mathcal{O}(D)$: sheaf of meromorphic functions f on C such that

f is hol. outside $\{p_1, \cdots, p_k\}$, and $\mathrm{ord}_{p_i}(f) + m_i \geq 0$

Riemann-Roch for C:

$$h^0(D) - h^0(K_C - D) = 1 - g + \deg(D).$$

$$h^0(D) = \dim \underbrace{H^0(C, \mathcal{O}(D))}_{\text{set of global sections of sheaf } \mathcal{O}(D) \text{ over } C}$$

K_C: canonical divisor of C ($\mathcal{O}(K_C)$ is canonical line bundle on C, sometimes write K_C for this line bundle)

Serre duality: $h^0(K_C - D) = h^1(D)$

$$(= \dim H^1(C, \mathcal{O}(D)))$$

$$h^0(D) - h^0(K_C - D) = h^0(D) - h^1(D) = \chi(\mathcal{O}(D))$$

$$1 - g = h^0(\mathcal{O}_C) - \underbrace{h^1(\mathcal{O}_C)}_{\substack{\| \\ h^{0,1} = g}} =: \chi(\mathcal{O}_C).$$

RR for C: $\chi(\mathcal{O}(D)) - \chi(\mathcal{O}) = \deg(D) \quad (= \sum m_i)$.

RR for vector bundles on C:

$\begin{array}{c} E \\ \downarrow \\ C \end{array}$ holomorphic vector bundle, $r = \mathrm{rank}(E)$

$\mathcal{O}(E)$: sheaf of holomorphic sections of E.

Thm: $\chi(\mathcal{O}(E)) - \chi(\mathcal{O}_C^r) = \deg(E)$.

$$(= \deg(\Lambda^r E)).$$

here: $\chi(\mathcal{O}_C^r) = r\,\chi(\mathcal{O}_C) = r(1-g)$.

Hirzebruch-Riemann-Roch for holo. vector bundle E on a compact complex manifold X, $n = \dim_{\mathbb{C}}(X)$.

$$\chi(X, E) = h^0(X, E) - h^1(X, E) + \cdots + (-1)^n h^n(X, E).$$

Thm (HRR):

$$\chi(X, E) = \int_X ch(E)\, td(X)$$

here, $ch(E)$: chern charater of E.

$td(X)$: Todd class of tangent bunlle of X

$ch(E), td(X) \in H^*(X, \mathbb{Q})$.

· Chern class of complex vector bundle over a smooth manifold

$$\begin{array}{ll} E & C_1(E), \ C_2(E), \ \cdots \\ \downarrow & \\ X & C(E) = 1 + C_1(E)\, t + C_2(E)\, t^2 + \cdots \end{array}$$

$1 \to E_1 \to E \to E_2 \to 1$ exact.

$$C(E) = C(E_1) \cdot C(E_2).$$

Define chern charater as follows:

for line bundle L,

$$ch(L) = \exp(C_1(L)) := \sum_{m=0}^{\infty} \frac{C_1(L)^m}{m!}$$

$V = L_1 \oplus L_2 \oplus \cdots \oplus L_n$,

define $ch(V) = ch(L_1) + ch(L_2) + \cdots + ch(L_n)$

we have: $$ch(V) = \sum_{m, i} \frac{C_1(L_i)^m}{m!}$$

the factor of $ch(V)$ in $H^{2m}(X, \mathbb{Q})$ is

$\frac{1}{m!} \sum_{i=1}^{n} C_1(L_i)^m$, can be expressed as a polynomial of elementary symmetric polynomials in $c_1(L_1), \dots, c_1(L_n)$

$$c_1(L_1)+c_1(L_2)+\cdots+c_1(L_n) = c_1(V)$$

$$\sum_{1\le i,j\le n} c_1(L_i)\, c_1(L_j) = c_2(V), \qquad \sum c_1(L_i)\, c_1(L_j)\, c_1(L_k) = c_3(V)$$

$\dots$

$$c(V) = c(L_1)\cdots c(L_n)$$
$$= (1+c_1(L_1)t)\cdots(1+c_1(L_n)t)$$

$$c_2(V) = \sum_{1\le i<j\le n} c_1(L_i)\, c_1(L_j)$$

Then we can generalize the definition of Chern character to general complex vector bundles.

ch satisfies: $ch(V\oplus W) = ch(V) + ch(W)$

$ch(V\otimes W) = ch(V)\, ch(W)$.

ch is a ring-homomorphism from $\underbrace{K(X)}_{(K\text{-group of } X)}$ to $H^*(X,\mathbb{Q})$.

<u>Todd (X)</u> for X compact complex manifold.

$$Q(x) = \frac{x}{1-e^{-x}} = 1+\frac{x}{2}+\sum_{i=1}^{\infty}\frac{B_i}{(2i)!}x^{2i}$$

$$= 1+\frac{x}{2}+\frac{x^2}{12}-\frac{x^4}{720}+\cdots$$

B_i: called Bernoulli number.

$$E \to X \qquad td(E) := \prod_{i=1}^{n} Q(\alpha_i) = \text{a polynomial in } c_1, c_2, \cdots, c_n \in H^0(X,\mathbb{Q}) \oplus H^2(X,\mathbb{Q}) \oplus H^4(X,\mathbb{Q}) \oplus \cdots$$

$\dim_{\mathbb{C}} X = n$

$\alpha_1, \cdots, \alpha_n$ called Chern roots, satisfying:

$$\alpha_1 + \cdots + \alpha_n = c_1(E)$$

$$\sum \alpha_i \alpha_j = c_2(E)$$

$$\vdots$$

$$\alpha_1 \cdots \alpha_n = c_n(E)$$

$td(E) =$

$Q(\alpha_1) \cdots Q(\alpha_n)$ $\qquad Q(x) = 1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \cdots$

constant term $= 1$

degree 1: look at $(1+\frac{\alpha_1}{2})(1+\frac{\alpha_2}{2}) \cdots (1+\frac{\alpha_n}{2})$

$$\longrightarrow \frac{1}{2}(\alpha_1 + \cdots + \alpha_n) = \frac{c_1}{2}$$

<u>degree 2</u>: look at

$$(1+\frac{\alpha_1}{2}+\frac{\alpha_1^2}{12}) \cdots (1+\frac{\alpha_n}{2}+\frac{\alpha_n^2}{12})$$

$$\rightsquigarrow \frac{1}{12}(\alpha_1^2 + \cdots + \alpha_n^2) + \frac{1}{4}\sum_{1 \le i,j \le n} \alpha_i \alpha_j$$

$$= \frac{1}{12}(\alpha_1 + \cdots + \alpha_n)^2 - \frac{1}{6}\sum \alpha_i \alpha_j + \frac{1}{4}\sum \alpha_i \alpha_j$$

$$= \frac{1}{12}\left(\sum \alpha_i\right)^2 + \frac{1}{12}\sum \alpha_i \alpha_j$$

$$= \frac{1}{12}c_1^2 + \frac{1}{12}c_2$$

$$td(E) = 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} + \frac{c_1 c_2}{24} + \frac{-c_1^4 + 4c_1^2 c_2 + c_1 c_3 + 3c_2^2 - c_4}{720} + \cdots$$

HRR: $\chi(E) = \int_X ch(E)\, Td(TX)$.

$$ch(E) = \text{rank}(E) + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}[c_1^3 - 3c_1c_2 + 3c_3] + \cdots$$

Assume X a compact complex surface.

$$td(TX) = 1 + \frac{c_1(TX)}{2} + \frac{c_1(TX)^2 + c_2(TX)}{12}$$

$$= 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12}$$

$$ch(E) = r + c_1(E) + \frac{1}{2}(c_1(E)^2 - 2c_2(E))$$

HRR: $\chi(E) = \int_X ch(E)\, td(TX)$

$$= \int_X \left\{ \frac{1}{2}(c_1(E)^2 - 2c_2(E)) + \frac{1}{2} c_1(E) c_1 + \frac{r}{12}(c_1^2 + c_2) \right\}$$

Assume $E = \mathcal{O}(D)$ a line bundle on X associated to a divisor D on X.

$r = \text{rank}(E) = 1.$

$$\chi(E) = \int_X \left[\frac{1}{2}(c_1(E)^2 - 2c_2(E)) + \frac{1}{2} c_1(E) c_1 + \frac{1}{12}(c_1^2 + c_2)\right]$$

$$= \frac{1}{2} c_1(E)^2 - c_2(E) + \frac{1}{2} c_1(E) c_1 + \frac{1}{12}(c_1^2 + c_2)$$

If we take $D = 0$, i.e. $E = \mathcal{O}_X$

then $\chi(\mathcal{O}_X) = \frac{1}{12}(c_1^2 + c_2)$. [Noether formula].

$$\chi(E) = \frac{1}{2} c_1(E)^2 - c_2(E) + \frac{1}{2} c_1(E)\, c_1 + \chi(\mathcal{O}_X)$$

$$= \chi(\mathcal{O}_X) + \frac{1}{2}(D\cdot D - D\cdot K),$$

HRR for line bundle on complex surface S $\Longleftrightarrow$

$$\begin{cases} \text{Noether formula} \quad \chi(\mathcal{O}) = \frac{1}{12}(c_1^2 + c_2) \\ \chi(D) = \chi(\mathcal{O}) + \frac{1}{2}(D^2 - D\cdot K), \quad D \text{ a divisor on} \end{cases}$$

Adjunction formula

M: complex manifold, K_M: canonical line bundle on M.

$\mathbb{C}P^n$ $\quad \mathcal{O}(-1) \downarrow \mathbb{C}P^n$: $\underbrace{\text{tautological}}_{\text{典则}}$ line bundle.

$\mathcal{O}(m) \downarrow \mathbb{C}P^n$ global sections = polynomials of degree m.

$m \geq 1$

$\mathcal{O}(m)$, $m \in \mathbb{Z}$ are all line bundles on $\mathbb{C}P^n$.

$K_{\mathbb{C}P^n} = \mathcal{O}(-n-1)$ (exercise).

$Z_0 \cdots Z_n$. $\quad \underline{Z_0 \neq 0}$. $\quad z_i = \frac{Z_i}{Z_0}$.

$dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$ on $Z_0 \neq 0$.

$\underline{Z_1 \neq 0}$. $\quad w_i = \frac{Z_i}{Z_1}$, $\quad i = 0, 2, 3, \cdots, n$.

$dz_1 \wedge \cdots \wedge dz_n = ? \cdot dw_0 \wedge dw_2 \wedge \cdots \wedge dw_n$.

$\uparrow \quad \overline{\omega_0^{n+1}}$

$\rightsquigarrow \quad K_{\mathbb{CP}^n} = \mathcal{O}(-n-1)$.

Adjunction. $\quad Y \overset{j}{\hookrightarrow} X \quad$ compact complex manifolds.

K_Y, K_X relation ?

I_Y : ideal sheaf of Y.

$\underbrace{I_Y / I_Y^2}_{\text{restriction of } I_Y \text{ to } Y} \longrightarrow j^*\Omega_X \longrightarrow \Omega_Y \quad$ exact.

Ω_X : sheaf of holo. 1-forms.

$$\det(\Omega_Y) = \det(j^*\Omega_X) \otimes \det(I_Y / I_Y^2)^\vee$$

$$K_Y = K_X|_Y \otimes \det(I_Y / I_Y^2)^\vee$$

If $\dim Y = \dim X - 1$.

$I_Y = \mathcal{O}_X(-Y)$, we have:

$$K_Y = (K_X \otimes \mathcal{O}_X(Y))|_Y.$$

adjunction formula.

$D \subset S$. $\quad D$: Smooth curve, S: compact complex surface.

$0 \to \mathcal{O}(-D) \longrightarrow \mathcal{O}_S \longrightarrow j_*\mathcal{O}_D \to 0$ exact

$\quad \otimes K \otimes \mathcal{O}(D)$

$0 \to K \longrightarrow K \otimes \mathcal{O}(D) \to \underbrace{K \otimes \mathcal{O}(D)|_D}_{} \to 0$.

$$\begin{array}{c}\shortparallel\\ K_D\end{array}$$

$$\chi(K_D) + \chi(K_S) = \chi(K \otimes \mathcal{O}(D)).$$

$$\chi(K_D) = \underset{h^{1,0}(D)}{\underset{\shortparallel}{h^0(K_D)}} - \underset{h^{1,1}(D)}{\underset{\shortparallel}{h^1(K_D)}} = g-1$$

D

g holes

So $\quad g-1 = \chi(K \otimes \mathcal{O}(D)) - \chi(K)$

$$\overset{RR}{=} \chi(\mathcal{O}) + \frac{1}{2}\left((K+D)^2 - (K+D)\cdot K\right)$$

$$- \left[\chi(\mathcal{O}) + \frac{1}{2}\left(K^2 - K^2\right)\right]$$

$$= \frac{1}{2}\left((K+D)^2 - (K+D)\cdot K\right)$$

$$= \frac{1}{2}\left(D^2 + D\cdot K\right)$$

So: $\quad g(D) = 1 + \frac{1}{2}\left(D^2 + D\cdot K\right).$

genus formula for curves in S.