Today: complex projective surface.

- Enriques - Kodaira classification.

complex surface: complex manifold of dimension 2

A complex projective surface is a compact complex surface that can be embedded holomorphically into $\mathbb{C}P^N$.

E.g. hypersurfaces of $\mathbb{C}P^3$,

complete intersections in $\mathbb{C}P^N$ of dim 2

Through this lecture, we use S to denote a complex surface.

S can be regarded as a compact 4-mfd.

we have: $\varphi: H^2(S,\mathbb{Z})_f \times H^2(S,\mathbb{Z})_f \longrightarrow \mathbb{Z}$.

$$\alpha, \beta \longmapsto \int_S \alpha \cup \beta \in \mathbb{Z}.$$

free part of $H^2(S,\mathbb{Z})$

This is a $\underbrace{\text{non-degenerate}}_{\text{by Poincaré duality}}$, bilinear form.

it is unimodular (i.e. can find a $\mathbb{Z}$-basis $e_1 - e_k$ of $H^2(S,\mathbb{Z})_f$, such that the intersection matrix

$$\begin{bmatrix} \varphi(e_1,e_1) & \varphi(e_1,e_2) & \cdots \end{bmatrix}$$ has $\det = \pm 1$.

$$\begin{pmatrix} & \cdots & \\ & \varphi(e_i, e_j) & \end{pmatrix}_{k\times k}$$

This is equivalent to:

Any group homomorphism from $H^2(S,\mathbb{Z})_f$ to $\mathbb{Z}$

is of the form $\varphi(\alpha, \cdot)$: $\beta \longmapsto \varphi(\alpha,\beta)$.

$\alpha, \beta \in H^2(S,\mathbb{Z})$, we sometimes simply denote $\varphi(\alpha,\beta)$ by

$\alpha\cdot\beta$, and called it the intersection number.

Definitions & Notation.

M : complex manifold.

Dolbeault Cohomology :

$$H^{p,q}(M) := \frac{\bar\partial\text{-closed smooth differential forms of type } (p,q)}{\bar\partial\,(\text{smooth diff. forms of type } (p,q-1))}$$

Ω^p : sheaf of holo. p-forms on M.

sheaf cohomology $H^q(M, \Omega^p)$ (see Hartshorne, GTM 52)

Dolbeault Isomorphism: $H^q(M,\Omega^p) \xrightarrow{\cong} H^{p,q}(M)$.

[use a fine resolution $\Omega^p \xrightarrow{\bar\partial} \mathcal{A}^{p,0} \xrightarrow{\bar\partial} \mathcal{A}^{p,1} \xrightarrow{\bar\partial} \cdots$)

We define $h^{p,q}(M) = \dim_{\mathbb{C}} H^{p,q}$.

For complex surfaces, we have the following Hodge diamond:

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array}$$

When S is compact, we have equalities:

$$h^{0,0} = h^{2,2} = 1, \quad h^{1,0} = h^{1,2}, \quad h^{0,1} = h^{2,1}$$

(by Poincaré duality)

enough to look at:

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \end{array}.$$

When S is compact Kählerian surface,

then we have natural isomorphisms:

$$H^k(S, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q} \quad \text{(Hodge decomposition)}$$

$=$

such that $\overline{H^{p,q}} = H^{q,p}$ as sub. vector spaces of $H^k(S, \mathbb{C})$.

In particular, $h^{p,q} = h^{q,p}$.

enough to look at $h^{0,0} = 1$, $\underset{h^{0,1}}{\overset{h^{1,0}}{\shortparallel}}$, $\underset{h^{0,2}}{\overset{h^{2,0}}{\shortparallel}}$, $h^{1,1}$.

A complex proj. surface is automatically Kählerian thanks to the Fubini-Study metric.

X topological space, denote by $b_i(X) = \dim_{\mathbb{R}} H(X, \mathbb{R})$ the betti numbers.

for M complex manifold, we usually denote by

$$q = h^{1,0} \text{ and call it irregularity of } M$$

If $q = 0$, we call M regular.

M complex manifold, K_M: canonical line bundle

$P_n = \dim_{\mathbb{C}} H^0(M, K_M^n)$ called n-th plurigenus of M

S surface, $P_1 = h^0(S, K_M) = h^0(S, \Omega^2_S) = h^{2,0}$.

k: Kodaira dimension of M.

$k = -\infty$ if $P_n = 0$, $\forall n$.

otherwise, k is the minimal integer such that

$\left(\frac{P_n}{n^k}\right)_n$ is bounded.

Fact: $k = -\infty$ or $k \in \{0, 1, \dots, \dim M\}$ for

M a complex algebraic variety

[Riemann Roch]

S $\overset{\text{compact}}{\text{complex}}$ surface, $\kappa \in \{-\infty, 0, 1, 2\}$.

for complex vector bundles E over a smooth manifold M

Chern classes $c_i(E) \in H^{2i}(M, \mathbb{Z})$.

Chern polynomial $c(E) = 1 + c_1 t + c_2 t^2 + \cdots$

L, L' complex line bundles.

$c(L) = 1 + c_1(L)\, t$

* $c_1(L \otimes L') = c_1(L) + c_1(L')$.

* $1 \to E_1 \to E \to E_2 \to 1$ exact sequence

then $c(E) = c(E_1)\, c(E_2)$.

M: compact complex manifold of dimension n.

$E \to M$ complex vector bundle.

$c_1(E), \cdots, c_n(E)$. [$c_k(E) \in H^{2k}(M) = 0$ if $k > n$)

partition of n $\to$ $\underline{c_{i_1} c_{i_2} \cdots c_{i_l}} \in H^{2n}(M) \xrightarrow{\int_M} \mathbb{Z}$

$i_1 + \cdots + i_l = n$

S: compact complex surface,

TS : holo. tangent bundle.

$K = (\Lambda^2 TS)^\vee$.

$c_1(TS) =: c_1(S)$ $\qquad$ $c_2(S) = c_2(TS) = e(S)$ euler charact. class.

$\quad \| $

$-c_1(K)$

$$= b_0 - b_1 + b_2 - b_3 + b_4.$$

$$= 2 - 2b_1 + b_2.$$

Enriques - Kodaira classification:

$\kappa = -\infty$, i.e. $P_n = 0$, $\forall n$.

$(\Leftrightarrow P_{12} = 0)$

$q = 0$: S is rational (i.e. $S \dashrightarrow \mathbb{P}^2$ birational map)

$q \geq 1$. $Alb : S \longrightarrow \underbrace{Alb(S)}_{\text{complex torus of dimension } q}$

$\rightsquigarrow$ S is a ruled surface.

M compact complex manifold,

$H^0(M, \Omega^1)$: space of global holomorphic 1-forms.

$H_1(M, \mathbb{Z}) \longrightarrow H^0(M, \Omega^1)^\vee$

$\gamma \longmapsto \int_\gamma \cdot$; $\omega \mapsto \int_\gamma \omega$

Assume M Kähler, $h^{0,1} = h^{1,0}$, free abelian then $H_1(M,\mathbb{Z})$ is mapped to a subgroup of rank

$b_1 = h^{1,0} + h^{0,1}$,

$H^0(M,\Omega^1)^\vee / H_1(M,\mathbb{Z}) =: \mathrm{Alb}(M)$ is a complex torus.

$$\mathrm{Alb}: M \longrightarrow \mathrm{Alb}(M) \qquad \underline{p \in M \text{ fixed}}.$$

$$q \longmapsto \left[\int_\gamma \cdot\right]$$

γ: a path from p to q.

well-defined in $\mathrm{Alb}(M)$

$K=2$, S is called of general type.

- $c_1^2, c_2 > 0$
- Miyaoka-Yau inequality: $c_1^2 \le 3c_2$.
  (1977, proved independently by –, –)
- (Max) Noether Inequality: $5c_1^2 - c_2 + 36 \ge 0$.
- $c_1^2 + c_2 \equiv 0 \pmod{12}$

Can construct coarse moduli spaces, but only know a precise characterization for a small number of cases.

$K=1$, S must be elliptic surface

(i.e. $\exists$ $S \longrightarrow B$ surjective, holomorphic
curve map.

such that all but finitely many fibers has genus one.

$\mathcal{J}$

$\underline{K=0} \Leftrightarrow (P_n)$ bounded.

Hirzebruch - Riemann - Roch for surface $\Rightarrow$

Noether formula:

$$12\chi(\mathcal{O}_S) = C_2 + C_1^2 \quad \text{———} \quad (*)$$

$$\chi(\mathcal{O}_S) = h^0(\mathcal{O}_S) - h^1(\mathcal{O}_S) + h^2(\mathcal{O}_S)$$

$$= 1 - h^{0,1} + h^{0,2}.$$

Suppose S is a compact Kähler Surface,

then $h^{0,1} = h^{1,0} = q$, $h^{0,2} = h^{2,0} = P_1$,

$$C_2 = e(S) = 2 - 2b_1 + b_2 = 2 - 4q + 2P_1 + h^{1,1}$$

When $K=0$, we have $C_1^2 = 0$. prove later

Use $(*)$: $12(1-q+P_1) = 2 - 4q + 2P_1 + h^{1,1}$

$$\Rightarrow 10 - 8q + 10P_1 - h^{1,1} = 0. \quad \text{———} \quad (2)$$

$$P_1 = h^0(K_S) = \begin{cases} 1 & \text{when } K_S \cong \mathcal{O}_S \end{cases}$$

$$\begin{cases} \quad 0 & \text{otherwise} \end{cases}$$

If $K_S \cong \mathcal{O}_S$, then ② $\Rightarrow$ $8q + h^{1,1} = 20$

$\Rightarrow$ $q = 0$, $h^{1,1} = 20$ [K3 surface]

$q = 1$, $h^{1,1} = 12$. not exist.

$q = 2$, $h^{1,1} = 4$ torus of dim 2. $\mathbb{C}^2/\Lambda$.

always Kählerian, not nec. algebraic

If $K_S \not\cong \mathcal{O}_S$, then $P_1 = 0$

② $\Rightarrow$ $8q + h^{1,1} = 10$

$\Rightarrow$ $q = 0$, $h^{1,1} = 10$ Enriques surface.

$q = 1$, $h^{1,1} = 2$ hyperelliptic surface.

$\Big\}$ always algebraic

finite quotient of a product of two elliptic curves.

Beauville: complex algebraic surfaces,

Barth, etc.: compact complex surfaces