Lecture 10 13 April 2022

Goal: Relate I_M to Green's function.

Recall: $(\Gamma, r) \rightsquigarrow (\Gamma_n, r_n)$ sequence of finite networks that exhaust Γ.

$\rightsquigarrow (\Gamma_n', r_n)$ finite networks by shorting all vertices in $V - V_n$

$V_n' = V_n \cup \{b_n\}$.

For each (Γ_n', r_n), there exists unique soln I_n'

$$\partial I_n' + z_n = 0$$

$$\langle I_n', z \rangle = 0 \quad \text{for all finite } z \text{ in } \Gamma_n'$$

$\Rightarrow$ Previous result: $\lim_{n \to \infty} \| I_n' - I_M \|_{H_1} = 0$; $\lim_{n \to \infty} W(I_n') = W(I_M)$

Corollary: $\Sigma \in \partial H_1 \iff \{W(I_n')\}_{n\in\mathbb{Z}_+}$ is bounded

(i.e. $\exists\ \Xi \in H_1$ s.t. $\partial\Xi = \Sigma$)

Pf: $(\Longrightarrow)$ is easy.

If $\Sigma \in \partial H_1$, $\Rightarrow \lim W(I_n') = W(I_m)$

$\Rightarrow \{W(I_n')\}$ is bounded (above by $W(I_m)$).

$(\Longleftarrow)$. Suppose $\{W(I_n')\}$ is bounded.

(by sequential compactness of H_1^{**}).

$\Rightarrow$ There is a converging subsequence, that converge weakly to ~~and~~ $\underline{I \in H_1}$

s.t. $\lim\limits_{n\to\infty} \| I_n' - I \| = 0$

(Ex)

$\Rightarrow$ We have pointwise convergence

$$I(xy) = \lim_{n\to\infty} I_n'(xy).$$

check:

$\Rightarrow$

$$\partial I + 2 = 0$$

$$\langle I, z \rangle = 0 \qquad \text{for all } z \in Z^*.$$

Now we express I_n' in terms of Green's function $G(x,y;b_n)$.

($\Leftrightarrow$ Consider (Γ_n', r_n)).

Well defined since b_n as the absorbing vertex.

Write: $G_n'(x,y) := G(x,y;b_n)$ in (Γ_n', r_n).

Exercise: $\lim_{n\to\infty} G_n'(x,y) = G(x,y)$ in (Γ, r).

(idea: push b_n to ∞).

For each (Γ_n', r_n)

$$\partial I_n' + \eta_n = 0 \quad \Rightarrow \quad (I - P_n) u_n = f_n$$

$\langle I_n', z \rangle = 0$ for z finite cycle.

where $f_n(x) = -\dfrac{\eta_n(x)}{c(x)}$.

$$\Rightarrow \quad u_n(x) = \sum_{y \in V} G_n'(x, y) f_n(y) \quad \text{and } u_n(b_n) = 0.$$

$$\Rightarrow I_n'(xy) = c(x,y)\big(u_n(x) - u_n(y)\big)$$

$$W(I_n') = \frac{1}{2} \sum_{xy \in E} r(xy) \left(I_n'(xy)\right)^2$$

$$= \frac{1}{2} \sum_{xy \in E} c(xy) \left(u_n(x) - u_n(y)\right)^2$$

$$= \sum_{x \in V_n} \sum_{y \in V_n(x)} c(xy)\, u_n(x) \left(u_n(x) - u_n(y)\right)$$

$$= \sum_{x \in V_n} u_n(x) \left(\sum_{y \in V(x)} c(xy) \left(u_n(x) - u_n(y) \right) \right)$$

$$= \sum_{x \in V_n} u_n(x)\, c(x) \left(\left(Id - P_n \right) u_n \right)(x).$$

$$= \sum_{x \in V_n} c(x)\, u_n(x)\, f_n(x)$$

$$= \sum_{x,y \in V_n} c(x)\, G_n'(x,y)\, f(y)\, f(x).$$

Thus, $z \in \partial H$, $\iff W(I_n') = \sum_{x,y \in V_n} c(x)\, G_n'(x,y)\, f(y)\, f(x) < \infty$.

as $n \to \infty$.

Ch III Section 4. (P.43) Transient network.

Know: $G(x,y)$ exists and finite.

locally finite i.e. $c(x) := \sum_{y \in N(x)} c(xy) < \infty$.

<u>Thm</u> Suppose (Γ, r) transient. And z 0-chain with $f(x) := \frac{z(x)}{c(x)}$. A sufficient condition for $z \in \partial H$, is

$$(*) \qquad \sum_{x,y \in V} c(x)\, G(x,y)\, |f(y)|\,|f(x)| < \infty,$$

This condition is also necessary if $f \geq 0$.

If $(*)$ holds,

(1). $G(|f|)(x) < \infty$ for all $x \in V$,

(2). $\lim_{n \to \infty} G_n'(f)(x) = G(f)(x)$.

(3). $G(f)$ is the potential of I_M.

(4). $W(I_M) = \sum_{x,y \in V} c(x) G(x,y) f(x) f(y)$

(Remark:

For each n, $c(b_n) < \infty$.

However, $c(b_n) \to \infty$ as $n \to \infty$).

Corollary. If (Γ, r) transient, then for every $a \in V$,

$$\eta := -c(a)\,\delta_a \in \partial H_1.$$

Pf: $f(x) = -\dfrac{\eta(x)}{c(x)} = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \neq a. \end{cases}$

Check $(\ast)$, $\displaystyle\sum_{x,y \in V} c(x)\, G(x,y)\, f(x)\, f(y) = c(a)\, G(a,a) < \infty.$

$\Rightarrow \quad \eta \in \partial H_1.$

Cor. (Γ, r) is transient $\Leftrightarrow$ exists $I \in H_1$ s.t.

$\partial I + \delta_a = 0$ for some $a \in V$.

Pf ($\Leftarrow$) Existence of I $\Rightarrow$ $\delta_a \in \partial H_1$ $\Rightarrow$ $\lim_{n\to\infty} G(a,a,h_n) < \infty$

$\qquad = W(I_M) = G(a,a)$

$\Rightarrow$ ~~potential~~ $\cancel{u(a) = G(a,a) < \infty}$

$\Rightarrow$ ~~transient.~~ $\Rightarrow$ transient.

($\Rightarrow$) Proved already.

$\delta_a \in \partial H_1 \Rightarrow \exists E \in H_1$ s.t. $\partial E = \delta_a$.

$\Rightarrow$ Project E to Z^*.

$$E = I_M + K$$

$$H_1 = Z^* \oplus (Z^*)^\perp$$

$W(I_M) \le W(E) < \infty$ since $E \in H_1$

Effective resistance

$$(\Gamma, r) \rightsquigarrow (\Gamma_n, r_n) \rightsquigarrow (\Gamma_n', r_n), \qquad V_n' = V_n \cup \{b_n\},$$

Pick $a \in V$. Assume $a \in V_n$ for all n,

$$R_n(a, b_n) = u_n(a) - \cancel{u_n(b_n)} \to 0$$

$$= u_n(a)$$

$$= \frac{G_n'(a,a)}{c(a)}.$$

Recall

$$z_n = \delta_a - \delta_{b_n}$$

$$u_n(x) = \frac{G_n'(x,a)}{c(a)} - \frac{G_n'(x,b_n)}{c(b_n)}$$

$$= \frac{G_n'(a,b_n)}{c(a)}$$

$$R_{eff}(a) = R_{eff}(a, \infty) := \lim_{n\to\infty} R_n(a, b_n) = \frac{G(a,a)}{c(a)}$$

Corollary: (Γ, r) is transient $\Leftrightarrow$ $R_{eff}(a, \infty) < \infty$ for some vertex $a \in V$.

Corollary: (Γ, r) transient. know: $-c(a)\delta_a \in \partial H$,

potential $G(\cdot, a)$ of I_M.

and $G(\cdot, a) \in D_0$

where D_0 is the closure of functions with finite support in D $\Leftarrow$ space with finite energy.

Pf: For each n, using (Γ_n', r_n)

$$u_n(x) = \begin{cases} G_n'(x, \overset{a}{\cancel{b_n}}) & \text{for } x \in V_n \\ 0 & \text{for } x \notin V_n \end{cases}$$

$\Rightarrow$ u_n has finite support

and $\lim u_n(x) = \lim G_n'(x,a) = G(x,a)$

Rmk: $\lim\limits_{x\to\infty} G(x,a) \to 0$.

$$R_{eff}(a) = u(a) - \cancel{u(\infty)}^{\to 0} = \frac{G(a,a)}{c(a)}$$

Plan: (Mon) Recurrent network

(Wed) $D = D_0 \oplus HD.$

Finite graph: $z \in \partial H_1 \iff \sum_{x \in V} z(x) = 0.$

Claim: Infinite: Assume $\sum_{x \in V} |z(x)| < \infty$ and $\sum_{x \in V} z(x) = 0$

then $z \in \partial H_1$

E.g. $z = \delta_a - \delta_b$,

$\exists$ finite 1-chain $\Sigma^{\in H_1}$ s.t. $\partial \Sigma = z.$

$\Rightarrow z \in \partial H_1$