Lecture 9 11 April 2022

Z .... finite cycles

<u>Def</u>. ($Z \subset Z^*$) space of all 1-cycles, infinite or finite

$$Z \subset Z^*$$

Check: Z^* is closed in H_1.

Recall: H_1 Hilbert space of 1-chain with finite energy.

$$\partial_x : H_1 \to \mathbb{R} \quad \text{(know: continuous).}$$

$$I \mapsto (\partial I)(x) = \sum_{y \in V(x)} c(x,y)\, I(x,y)$$

$$Z^* = \bigcap_{x \in V} \operatorname{Ker}(\partial_x) \sim \text{countably many intersection of closed sets}$$

$$\Rightarrow Z^* \text{ closed.}$$

Thm Let $\tilde{Z}$ be any closed subspace of Z^* s.t. $Z \subset \tilde{Z}$.
Z finite cycles.

$(\; \cancel{Z} \subset \tilde{Z} \subset Z^* \;)$

Then for any $E \in H_1$, consider:

$$(*) \begin{cases} \partial \tilde{I} = 0 \\ (\tilde{I} - E, z) = 0 \quad \text{for } ~~\text{all finite cycles } z~~ \quad z \in \tilde{Z}. \end{cases}$$

$(*)$ has a unique soln in $\tilde{Z}$.

pf: Decompose $\underset{\Psi}{H_1} = \tilde{Z} \oplus \tilde{Z}^{\perp}$

$$E = \tilde{I} + K$$

$$- \begin{cases} \tilde{I} \in \tilde{Z} \Rightarrow \partial \tilde{I} = 0 \\ \langle E, z \rangle = \langle \tilde{I}, z \rangle \quad \text{for all finite cycles} \end{cases}$$

Thm, $\mathcal{Z} \subset \tilde{\mathcal{Z}}_1 \subset \tilde{\mathcal{Z}}_2 \subset \mathcal{Z}^*$.

Then, $W(\tilde{I}_1 - \bar{I}) \geq W(\tilde{I}_2 - \bar{I})$

Pf: Using $\tilde{\mathcal{Z}}_1$, $\Rightarrow \bar{I} = \tilde{I}_1 + K_1 \Rightarrow W(\bar{I}) = W(\tilde{I}_1) + W(\tilde{I}_1 - \bar{I})$

$\parallel$

Using $\tilde{\mathcal{Z}}_2$, $\Rightarrow \bar{I} = \tilde{I}_2 + K_2 \Rightarrow W(\bar{I}) = W(\tilde{I}_2) + W(\tilde{I}_2 - \bar{I})$

orthogonal decomposition

$$\tilde{\mathcal{Z}}_1 \subset \tilde{\mathcal{Z}}_2 \Rightarrow \tilde{\mathcal{Z}}_2 = \tilde{\mathcal{Z}}_1 \oplus H$$

$$\tilde{I}_2 = \tilde{I}_1 + \delta$$

$$W(\tilde{I}_2) = W(\tilde{I}_1) + W(\delta) \geq W(\tilde{I}_1).$$

$$W(\tilde{I}_1 - \bar{I}) - W(\tilde{I}_2 - \bar{I}) = W(\tilde{I}_2) - W(\tilde{I}_1) \geq 0$$

Def Let $\eta = \partial \bar{E}$ and $\bar{E} \in H_1$.

Consider $\tilde{I} \in Z^*$ s.t. it solves

$$\partial \tilde{I} = 0$$

$$\langle \tilde{I} - \bar{E}, z \rangle = 0 \quad \text{for all } z \in Z^*.$$

$\Rightarrow$ $I_M := \tilde{I} - \bar{E}$ minimal current. It solves

$$\partial I_M + \eta = 0$$

$$\langle I_M, z \rangle = 0 \quad \text{for all } z \in Z^*.$$

Remark: I_M depends only on 0-chain η (external currents at vertices) but not on $\bar{E}$.

Remark:

$$\begin{tikzcd} Z \arrow[r, phantom, "\subset"] & \hat{Z} \arrow[r, phantom, "\subset"] & Z^{*} \\ I_L := \hat{I} - E \arrow[u] & I_{\hat{Z}} := \hat{I} - E \arrow[u] & I_M := \hat{I} - E \arrow[u] \end{tikzcd}$$

need E finite.

makes no sense unless there canonical choice for E.

E.G. H_1 might or might not be finite

Remark: $Z = \partial E$ where E is finite

$$W(I_L) \geq W(I_M).$$

Question: Transient $\Rightarrow$ $G(x,y)$ is finite $\Rightarrow$ For suitable f 0-chain define $G(f)$

If well-defined, $G(f)$ is the potential of which current?

Ans: minimal current I_L. (To be done).

doesn't matter for I_M.

Def. $z = \delta_a - \delta_b$. $\Rightarrow$ $\exists$ ~~finite~~ 1-chain s.t. $\partial \Sigma = z$, $\Sigma \in H_1$

$\Rightarrow$ I_M minimal current

$\Rightarrow$ u potential of I_M s.t. $I_M(xy) = c(x,y)\, u(x) - u(y)$

$\Rightarrow$ (minimal) effective resistance.

$$R_{eff}(a,b) = u(a) - u(b) = W(I_M)$$

$$= \frac{1}{2}\sum_{xy \in E} r(xy)\,(I(xy))^2$$

$$= \| I_M \|^2_{H_1}$$

$$\Rightarrow \boxed{\partial I_M = -(\delta_a - \delta_b)}$$

Idea! $R_{eff} \sim I_M \sim$ Green's function $\sim$ transience of recurrence

Goal: Approximate I_m using finite sub-networks.

<u>Def</u>. (Γ, r) network.

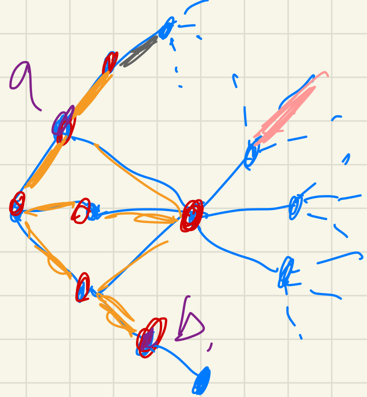
$U \subset V$ subset of vertices.

(Γ', r') obtained from (Γ, r) by shorting all vertices in $V - U$.

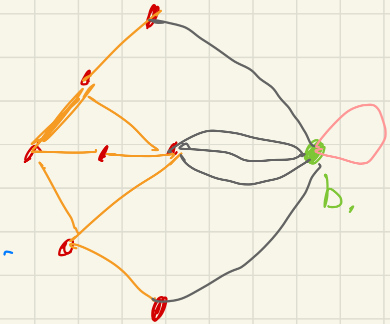
$V' := U \cup \{b\}$,

E' collection of edges in $xy \in E$ s.t. at least x, or $y \in U$.

$$r'(xy) := \begin{cases} r(xy) & \text{if } x, y \in U \\ \left(\sum_{\tilde{x}\tilde{y} \in \Phi^{-1}(xy)} \frac{1}{r(\tilde{x}\tilde{y})} \right)^{-1} & \text{if } x \in U \text{ but } y \notin U. \end{cases}$$



$$\left(\frac{1}{\tilde r} = \frac{1}{r_1} + \frac{1}{r_2} \right).$$



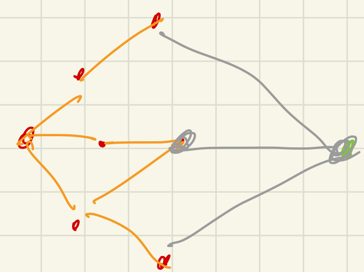
(Γ, r)

$U \subset V$

$\Downarrow$ simplify

$R_n'(a,b)$ in (Γ_n', r_n')

$\Rightarrow \quad \Phi : \Gamma = (V,E) \to \Gamma' = (V', E').$



Physically, replace all resistors not connected to U by zeros resistance.

(Γ', r')

(1). Consider $\Gamma_1 \subset \Gamma_2 \ldots \subset \Gamma_n \subset \ldots$ exhaustion of Γ

(sequence of finite subnetworks, $\bigcup_{n=1}^{\infty} \Gamma_n = \Gamma$.).

For each n,

Γ_n' which shorts all vertices in $V - V_n$.

(2). Given $v \in \partial H$,

Define restriction v to Γ_n' $\Rightarrow$ v_n.

$$v_n(x) = \begin{cases} v(x) & \text{if } x \in V_n. \\ -\sum_{y \in V_n} v(y) & \text{if } x = b_n. \end{cases}$$

(3). For each n, we find I_n on (Γ_n', r_n')

$$\partial I_n + z_n = 0$$

$$\langle I_n , z \rangle = 0 \quad \text{for all cycles } z \text{ in } (\Gamma_n', r_n').$$

$\Rightarrow$ I_n exists and unique

Thm. Assume $z \in \partial H_1$. Denote I_n currents in (Γ_n', r_n').

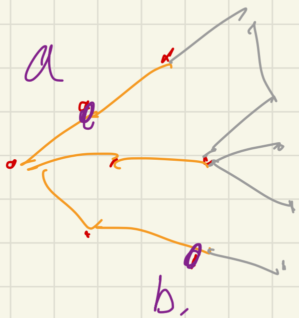
(P.40 in the book)

Then $\{W(I_n)\}_{n=1}^{\infty}$ is a bounded sequence.

and I_n converges to I_m in H_1,

Furthermore, $W(I_m) \leq I(J)$ for any J satisfying $\partial J + z = 0$.

Compare with I_L.



$R_n(a,b)$ in (Γ_n, r_n)

$$(\Gamma, r) \rightsquigarrow (\Gamma_n, r_n).$$

Physically, replace all external resistors by ∞ resistance

<u>Exercise</u>: $R'_n(a,b) \le R'_{n+1}(a,b) \le R^M_{eff}(a,b) \le R_{n+1}(a,b) \le R_n(a,b)$.

$$\lim R_n'(a,b) = R^M_{\text{eff}}(a,b)' \le \lim R_n(a,b) = R^L_{\text{eff}}(a,b)$$

$$\shortparallel$$

$$R_{\text{eff}}(a,b)$$

limit effective resistance

$$\mathbb{Z} \subset \mathbb{Z}^2 \subset \mathbb{Z}^3 \ \cancel{\subset W} \subset \mathbb{H}_1$$

$$W(I_m) \le W(I_L)$$

$$\shortparallel \qquad\qquad \shortparallel$$

$$D(u_m) \qquad D(u_L)$$

$$((I-P)\,u_M)(x) = -\frac{\eta(x)}{c(x)}$$

$u_L - u_M$ is harmonic,

To be done: " $\lim_{x\to\infty} u_M(x) = 0$ " for suitable η.