Theorem:

(i) In a ring category with left duals, the unit object is simple

(ii) — multi — semisimple.

all objects have left-dual

Rk: (i) ⇒ (ii), so we just need to prove (i).

- we will use (i) to show that some ring cat. is not left-rigid
- — in the next corollary. (when the unit is not simple).

---- (Summary of last remedier session !g) ---

Lemma: $X \xrightarrow{p} Y$ epi. Then X simple $\Rightarrow Y \simeq 0$ or X.

Lemma: Let $X \xrightarrow{f} Y$ nonzero morphism. Then

(i) X simple $\Rightarrow$ f mono.

(ii) Y simple $\Rightarrow$ f epi.

Corollary: In a ring category with left duals, the evaluation maps $(X^* \otimes X \xrightarrow{ev_X} 1)$ are epi., and the coevaluation maps $(1 \xrightarrow{coev_X} X \otimes X^*)$ are mono.

Corollary: Let $\mathcal{C}$ be a ring category with left duals, and $F: \mathcal{C} \to \mathcal{D}$ an exact k-linear monoidal functor into a multiring cat. $\mathcal{D}$.

Then F is faithful $\left(\begin{matrix} F(X)=0 \Rightarrow X=0 \\ F(f)=0 \Rightarrow f=0 \end{matrix} \right)$, so F is a tensor functor (by $\oplus$).

Prop: Let $\mathcal{C}$ be multiring cat. with left duals, $X \in \mathcal{C}_{ij}$, $Y \in \mathcal{C}_{jk}$ be nonzero objects. Then:

(1) $X \otimes Y \neq 0$.

(2) $\text{length}(X \otimes Y) \geq \text{length}(X) \cdot \text{length}(Y)$

(3) $X \otimes X^* \cong 1 \Rightarrow X$ invertible.

(4) If $\mathcal{C}$ is ring cat. then X invertible $\Rightarrow X$ simple.

Def: X is invertible if ev_X and $coev_X$ are iso.

Example: of a ring cat. $\mathcal{C}$ where the unit object is not semisimple: (to not simple)

triples (V, W, A) where V, W are finite dim. vector spaces (over $\Bbbk$ algebraically closed), and $A: V \to W$ linear map (i.e. a matrix), where

$$(V, W, A) \otimes (V', W', A') := (V \otimes V', W \otimes W', A \otimes A')$$

usual $\otimes$ of vector spaces; Kronecker product

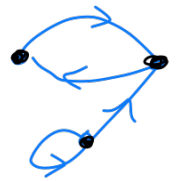
with obvious associativity constraints and unit $\mathbb{1} = (\Bbbk, \Bbbk, \mathrm{id})$

Rk: By above thm (i), $\exists$ object $X \in \mathcal{C}$ without left dual (idem for right)

Rk: Above category can be defined as the category of finite dim. rep. of the quiver of type A_2.

Def: A quiver is directed graph (meaning that you have vertices and oriented edges) where loops and multiple arrows are allowed.

Examples:



(A_2).

Def: A representation of a quiver is the assignment of a vector space to each vertex, and a linear map to each arrow (oriented edge) (going in the sense of the orientation).

Example: for A_2 

{ it is exactly the cat. of triple (V, W, A) as written above.

---- (Summary of last semester last session (20)) ----

Grothendieck rings and Frobenius-Perron dimensions

Let $\mathcal{C}$ be an abelian category. Recall that an object X has finite length if it has a filtration $0 = X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X$ $(n < \infty)$ which is Jordan-Hölder, i.e. X_i / X_{i+1} is simple for all i.

Theorem (J.H.). Let X be an object of finite length. Then, the sequence of simple objects $(S_i) = (X_i/X_{i+1})$ is independant of the choice of J.H. filtration. (up to iso and permutation), and any filtration of X can be completed into a J.H. one

. Let X be an object of finite length:

Def: Let $[X:S]$ be the multiplicity of S in (S_i) (up to iso)

↑ well-defined by above thm.

Lemma: Let X, X' be two objects of finite length (in an abelian cat.). Let S be a simple object. Then $X \simeq X' \Rightarrow [X:S] = [X':S]$.

Def: Let $\mathcal{C}$ be an abelian k-linear cat., where the objects have finite length. Let $Gr(\mathcal{C})$ be the Grothendieck group, i.e. the abelian group freely generated by the simple objects, denoted $(X_i)_{i\in I}$ up to iso.

$\forall X \in \mathcal{C}$, define $[X] \in Gr(\mathcal{C})$ as $[X] := \sum_{i\in I} [X:X_i]\, X_i$

Lemma: If $0 \to X \to Y \to Z \to 0$ is exact, then $[Y] = [X] + [Z]$

In particular, $[X \oplus Z] = [X] + [Z]$.

Multiplication on $Gr(\mathcal{C})$: Let $\mathcal{C}$ be a multiring cat (over k)

Let $(X_i)_{i \in I}$ be the simple objects of $\mathcal{C}$ (up to iso.).

we should also assume that all objects have finite length.

The tensor product $\otimes$ induces a multiplication on $Gr(\mathcal{C})$.

$$[X_i]\,[X_j] := [X_i \otimes X_j] = \sum_{k \in I} [X_i \otimes X_j : X_k]\, X_k .$$

called fusion rules. (But note that they do NOT completely characterize the category [even in the semisimple case] --> see categorification problem.)

Lemma: By extending above equality by linearity, we get $[X][Y] = [X \otimes Y]$ for all $X, Y \in \mathcal{C}$

(hint: use exactness of tensor product).

Lemma: Above multiplication is associative.

(hint: reduce to simple objects by linearity, and then use the associativity of $\otimes$ ...)

Corollary: $Gr(\mathcal{C})$ is a $\mathbb{Z}_+$-ring with unit $[1]$, called the Grothendieck ring of $\mathcal{C}$. (here $\mathbb{Z}_+ := \mathbb{Z}_{\geq 0}$)

Def: (i) A $\mathbb{Z}_+$-basis of a ring is a basis $B = (b_i)_{i \in I}$ s.t.

$$b_i b_j = \sum_{k \in I} c_{ij}^k b_k \quad , \text{ with } c_{ij}^k \in \mathbb{Z}_+$$

Notation: $[b_i b_j : b_k] = c_{ij}^k$

(ii) A $\mathbb{Z}_+$-ring is a ring with a fixed $\mathbb{Z}_+$-basis B and with $1 = \sum c_i b_i$ where $c_i \geq 0$.

(iii) A $\mathbb{Z}_+$-ring is called unital when $1 \in B$.

Rk: A $\mathbb{Z}_+$-ring is a free $\mathbb{Z}$-module, i.e. of the form $\mathbb{Z}^r$ (as $\mathbb{Z}$-module) where $r = |B|$ called the rank.

Def: A $\mathbb{Z}_+$-ring (with basis B) is called transitive (useful for the Frobenius-Perron thm) if $\forall X, Z \in B$, $\exists Y_1, Y_2 \in B$ s.t. $[XY_1 : Z] \neq 0$ and $[Y_2 X : Z] \neq 0$.

Prop: Let $\mathcal{C}$ be a ring category with left duals. Then $\mathrm{Gr}(\mathcal{C})$ is a transitive unital $\mathbb{Z}_+$-ring.

Let A be a transitive unital $\mathbb{Z}_+$-ring of finite rank. Let $X, Y, Z \in B$

Def (fusion matrix): Define M_X to be the fusion matrix given by left multiplication and transpose: $M_X := ([XY : Z])_{Z,Y}$

Def: (Frobenius-Perron dimension) $\boxed{FPdim(X) := \|M_X\|}$ (matrix norm)

Rk: We needed transpose to make (M_X) a representation of the ring, because:

Associativity $\iff (X_i X_j) X_k = X_i (X_j X_k)$

$$\iff \sum_s c_{ij}^s c_{sk}^t = \sum_s c_{jk}^s c_{is}^t \qquad \forall i,j,k,t$$

$$M_{X_i} = (c_{ij}^k)_{kj}$$

$$\iff M_{X_i} M_{X_j} = \sum c_{ij}^k M_{X_k} \quad \text{(i.e. representation)}.$$

Def: A fusion ring is a unital $\mathbb{Z}_+$-ring of finite rank with a bijection $()^*: B \to B$ s.t. $M_{X_i^*} = (M_{X_i})^*$ (Frobenius reciprocity)

(* on the right: matrix adjoint.)

See you next time.