Recall. If K/k is finite extention, suppose $K=k(a_1,\cdots,a_r)$, define

$$N_{a_1\cdots a_r/k}=N_{a_1/k}\circ N_{a_2/k(a_1)}\circ\cdots\circ N_{a_r/k(a_1,\cdots a_{r-1})}$$

Thm 3.7. The $N_{a_1,\cdots a_r/k}$ is independent of the choice of $a_1,\cdots,a_r$. Hence we obtain the norm map

$$N_{K/k}:K_*^M(K)\to K_*^M(k).$$

Proof: We do by induction on $[K:k]$ and prove the statement after localizing at each prime p. Choose L/k as in Prop 3.8. The ring $L'=L\otimes_k K$ is artinian and has finite many prime ideals $P_1,\cdots,P_m$. Let $e_i=l_{L'_{P_i}}(L'_{P_i})$. The proof consists of the following steps:

1) Use Prop 3.9 to show that the diagram (L' is a product of fields, i.e. $m>1$)

$$\begin{tikzcd} K_*^M(K) \arrow[r, "(e_i)"] \arrow[d, "N_{a_1,\cdots a_r/k}"'] & \bigoplus K_*^M(L_i') \arrow[d, "\sum \boxed{N_{L_i'/L}}"] \\ K_*^M(k) \arrow[r] & K_*^M(L) \end{tikzcd}$$

$L_i':=k(P_i)$ ✓

Use induction hypothesis since $[L_i':L]<[K:k]$.

2) When $m=1$, i.e. L' is a field. I claim there is a sequence of fields $L=E_0\subseteq E_1\cdots\subseteq E_n=L'$ s.t. $[E_{i+1}:E_i]=p$ and E_{i+1}/E_i is a normal extension.
By definition of L, $[L':L]$ is a power of p. Choose a Galois extension M/L, $M\supseteq L'$ M/L is Galois. Then $Gal(M/L)$ is a p-group, hence has a composition series

$$Gal(M/L)=G_1\supseteq\cdots\supseteq G_n=Gal(M/L') \quad (\underbrace{G_i/G_{i+1}}_{\text{has order } p} \text{ is normal})$$

p-groups has nontrivial center

So we set $E_i:=L'^{G_{n-i+1}}$.

3) Show that $\underbrace{N_{a_1,\cdots a_r/k}=N_{E_1/E_0}\circ\cdots\circ N_{E_m/E_{m-1}}}$, hence independent of the choice of $a_1\cdots a_r$. $N_{a_1/L}\circ N_{a_2/L(a_1)}$

$$\begin{tikzcd} & E_2=L' & \\ E_1 \arrow[ur] & & L(a_1) \arrow[ul] \\ & L=E_0 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

normal extension of deg $=p$

E.g. $r=m=2$. $N_{a_1,a_2/k}=N_{E_1/E_0}\circ N_{E_2/E_1}$.

If $a_1\in E_1$, $N_{a_2/\underset{\underset{E_1}{\parallel}}{L(a_1)}}=N_{E_2/E_1}$. $N_{a_1/L}=N_{E_1/E_0}$

If $a_1\notin E_1$, $N_{a_2/L(a_1)}=N_{E_2/L(a_1)}$, $N_{a_1/L}\circ N_{a_2/L(a_1)}=N_{a_1/L}\circ N_{E_2/L(a_1)}\underset{\text{by Prop 13}}{=}N_{E_1/L}\circ N_{a_1/E_1}$

$=N_{E_1/L}\circ N_{E_2/E_1}$ □

Thm 3.12 (Weil Reciprocity) For any algebraic function field L/k, we have

$$\sum_{v\in DV(L/k)} N_{k(v)/k}\circ\partial_v=0.$$

on $K_*^M(L)$.

Proof: The point of the proof is to show that for every finite extension E/F between algebraic function fields and $w\in DV(F/k)$, we have

$$\begin{tikzcd} K_*^M(E) \arrow[r, "\partial_v"] \arrow[d, "N_{E/F}"'] \arrow[dr, phantom, "\circlearrowright"] & \bigoplus_{\substack{v\in DV(F/k)\\ v \text{ lies over } w}} K_{*-1}^M(k(v)) \arrow[d, "\sum N_{k(v)/k(w)}"] \\ K_*^M(F) \arrow[r, "\partial_w"] & K_{*-1}^M(k(w)) \end{tikzcd}$$

(Use Prop A)

Since L is a finite extension over $k(t)$, we have

$$\begin{tikzcd} L \arrow[r, squiggly] & C \arrow[r] \arrow[d, "\text{finite}"'] & \text{nonsingular complete alg. curve} \\ & \mathbb{P}^1\ni x & \end{tikzcd}$$

$\pi^{-1}(x)$

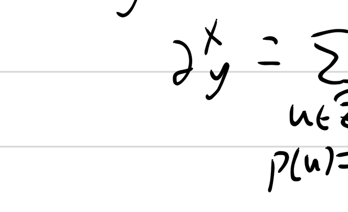
$$\sum_{v\in DV(L/k)}N_{k(v)/k}\circ\partial_v=\sum_{w\in\mathbb{P}^1_k}\sum_{v/w}N_{k(v)/k}\circ\partial_v$$

$$=\sum_{w\in\mathbb{P}^1_k}N_{k(w)/k}\circ\partial_w\circ N_{L/k(t)}$$

$=0$ by homotopy invariance (Thm 2.4). □

$$\sum_{w\in\mathbb{P}^1_k}N_{k(w)/k}\circ\partial_w=0$$

Def 3.13 Suppose X is an integral scheme. Define $K_*^M(X)$ by the exact sequence



$$0\to K_*^M(X)\to K_*^M(k(X))\xrightarrow{\boxed{\partial_y^{\xi_X}}}\bigoplus_{y\in X^{(1)}}K_{*-1}^M(k(y)).$$

ξ_X generic pt of X

$$K_1^M\quad k(X)^\times\xrightarrow{\text{ord}}\bigoplus\mathbb{Z}$$

Here, for any $x\in X^{(n)}$, $y\in X^{(n+1)}\cap\overline{x}$, we define $\partial_y^x:K_*^M(k(x))\to K_{*-1}^M(k(y))$ as the following: Let $Z=\overline{x}$ and $p:\tilde{Z}\to Z$ be the normalization.

$$\begin{tikzcd} \tilde{Z} \arrow[r] & Z \\ u \arrow[u, phantom, "\in"] \arrow[r, mapsto] & y \arrow[u, phantom, "\in"] \end{tikzcd}$$

Define $$\partial_y^x=\sum_{\substack{u\in\tilde{Z}\\ p(u)=y}}N_{k(u)/k(y)}\circ\partial_u.$$

M. Rost Chow groups with coefficients

Def 3.14 For any scheme X, define the Rost "complex"

$$C^p(X,K_n^M)=\bigoplus_{x\in X^{(p)}}K_{n-p}^M(k(x)).\qquad p>n,\ C^p(X,K_n^M)=0$$

Define $d_X:C^p(X,K_n^M)\to C^{p+1}(X,K_n^M)$ by ∂_y^x (in Def 3.13) for every $x\in X^{(p)}$, $y\in X^{(p+1)}\cap\overline{x}$.

Rmk: $H^n(C^*(X,K_n^M))=CH^n(X)$

$$\begin{tikzcd} C^{n-1}(X,K_n^M) \arrow[r] \arrow[d, equal] & C^n(X,K_n^M) \arrow[r] \arrow[d, equal] & C^{n+1}(X,K_n^M) \arrow[d, equal] \\ \bigoplus_{x\in X^{(n-1)}}k(x)^\times \arrow[r] & \bigoplus_{x\in X^{(n)}}\mathbb{Z} \arrow[r] & CH^n(X) & 0 \arrow[r] & 0 \end{tikzcd}$$

We want to prove that $d_X\circ d_X=0$ so $C^*(X,K_n^M)$ is indeed a complex.

Def 3.15 Suppose $f:X\to Y$ is a proper morphism between finite type schemes over a field. Define

$$f_*:C^p(X,K_*^M)\to C^{p+d_Y-d_X}(Y,K_{*+d_Y-d_X}^M)$$

as the following: If $y=f(x)$ and $[k(x):k(y)]<\infty$, then $(f_*)_y^x:K_*^M(k(x))\to K_*^M(k(y))$
$\parallel$
$N_{k(x)/k(y)}$

Otherwise $(f_*)_y^x=0$.

Def 3.16 Suppose $f:Y\to X$ is a flat morphism between f.t./k. Define

$$f^*:C^p(X,K_*^M)\to C^p(Y,K_*^M)$$

as the following. For every $x\in X$ and $y\in Y_x^{(0)}$, define ($Y_x = f^{-1}(x)$)

$$(f^*)_y^x=\underbrace{l_{\mathcal{O}_{Y_x,y}}(\mathcal{O}_{Y_x,y})}_{\text{Artinian}}\left(K_*^M(k(x))\to K_*^M(k(y))\right),$$

otherwise $(f^*)_y^x=0$.

Prop 3.17 1) $d_Y\circ f_*=f_*\circ\boxed{d_X}$ ($f:X\to Y$ proper) 2) $d_Y\circ f^*=f^*\circ d_X$ ($f:Y\to X$ flat)

Proof: 1) We show that $\delta(f_*)=d_Y\circ f_*-f_*\circ d_X=0$. There are three cases to consider: ($y\in Y^{(p+d_Y-d_X)}$, $x\in X^{(p)}$)

a: $y\notin\overline{f(x)}$: $\delta(f_*)_y^x=0$ by definition.

b: $y=f(x)$: Base change to $\overline{x}\subseteq X_y\to\mathrm{Spec}(k(y))$ is a proper curve. Then use Thm 3.12.

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(k(y)) \arrow[r] & Y \end{tikzcd}$$

c: $y\in\overline{f(x)}^{(1)}$: Then $[k(x):k(f(x))]<\infty$. Now use the compatibility between ∂ and norms. (See the proof of Thm 3.12) Prop A

2) $f:Y\to X$ is flat. Define $\delta(f^*)=d_Y\circ f^*-f^*d_X$. Suppose $y\in Y^{(p+1)}$ and $x\in X^{(p)}$. The only nontrivial case is when $f(y)\in\overline{x}^{(1)}$. By normalization and localization at y and $f(y)$, reduce to the case $X=\mathrm{Spec}\,R$, R is DVR and $Y=\mathrm{Spec}\,S$, S is local of dim ≤1.

Then $$\delta(f^*)_y^x=\left(\sum_{w\in Y_{(1)}}l_{S_w}(\mathcal{O}_{Y_x,w})\cdot\partial_y^w\circ K_*^M(k(w)/k(x))\right)-l_S(S/m_RS)K_*^M(k(y)/k(f(y)))\circ\partial_{f(y)}^x$$

Suppose $\tilde{S}$ is the normalization of S. We have

$$\partial_y^w\circ K_*^M(k(w)/k(x))=\sum_{u\in\tilde{S}_{(1)}}N_{k(u)/k(y)}\partial_u^w\circ K_*^M(k(w)/k(x))\quad\text{by Prop 3.9}$$

$$=\sum_{u\in\tilde{S}_{(1)}}l(\tilde{S}_{(u)}/m_R\tilde{S}_{(u)})\cdot N_{k(u)/k(y)}\circ K_*^M(k(u)/k(f(y)))\circ\partial_{f(y)}^x$$

Then the question is reduced to computation of lengths.

$$l(\tilde{S}/m\tilde{S})=\sum\underbrace{[\deg of]}_{\text{extension of fibers}}\cdot l(\ ).$$ □