Lemma: Let $\mathcal{C}$ be an additive category. The sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C$$

is exact iff $\forall M \in \mathcal{C}$

$$0 \longrightarrow F_M(A) \xrightarrow{F_M(i)} F_M(B) \xrightarrow{F_M(j)} F_M(C)$$

is exact, where $F_M := \hom_{\mathcal{C}}(M, -)$.

Rk: One way means that the functor F_M is left exact.

Rk: Same lemma (and rk) for contravariant functor $G_M := \hom_{\mathcal{C}}(-, M)$.

Prop: Let $\mathcal{C}$ be a multiring category with left duals. Then, the (contravariant) functor $(-)^* : X \mapsto X^*$ is exact. (Same with right duals and ${}^*(-)$).

[last semester, the proof required 3 slides].

Now we need to recall the notions of projective / injective objects in an abelian category:

Rk: We already know that in an abelian category $\mathcal{C}$, the functors $F_X = \hom_{\mathcal{C}}(X,-)$ and $G_Y = \hom_{\mathcal{C}}(-,Y)$ are left-exact. (but not exact in general)

Def: An object P (in an abelian cat.) is called projective if F_P is exact.
______ I ______ injective __ G_I ______.

Def: Let $X \in \mathcal{C}$. A projective cover of X is a projective object $P(X) \in \mathcal{C}$ together with an epi. $p: P(X) \to X$ ("surjective", $\mathrm{Coker}(p)=0$, X is a quotient object of $P(X)$) which is universal in the sense that:

$$\begin{tikzcd} & P \arrow[dl, dashed, "h"'] \arrow[dr, "g"] \arrow[d, phantom, "\circlearrowright"] & \\ P(X) \arrow[rr, "p"] & & X \end{tikzcd}$$

in other words

if $\exists P$ projective and $g: P \to X$ epi then $\exists h: P \to P(X)$ epi., with $p \circ h = g$

Rk: If a projective cover exists, then it is unique (up to isomorphism).

Dually (arrow reversed...):

Def: Let $X \in \mathcal{C}$. An injective hull of X is an injective object $I(X) \in \mathcal{C}$ together with a mono. $i : X \to I(X)$ ("injective", $\ker(i) = 0$, subobject) which is universal in the sense that:

$$\begin{tikzcd} & I & \\ I(X) \arrow[ur, dashed, "h"] & & X \arrow[ll, "i"] \arrow[ul, "g"'] \arrow[ul, phantom, "\circlearrowleft", shift right=6] \end{tikzcd}$$

in other words

if $\exists I$ injective, and $g : X \to I$ mono. then $\exists h : I(X) \to I$ mono., with $h \circ i = g$.

Rk: Same ...

Recall that a semisimple (every object is a direct sum of finitely many simple objects) (abelian) cat. is called finite if

(i) it has only finitely many simple objects (up to iso.)

Let us provide the general def (i.e. without semisimple assumption):

Def: A $\mathbb{k}$-linear abelian category $\mathcal{C}$ is finite if

(i) see above ... ,

(ii) Every object has finite length, (recall that it is about Jordan–Hölder sequence)

(iii) The hom spaces are finite dim. (over $\mathbb{k}$).

(iv) There are "enough projectives", i.e every simple object has a projective cover.

Prop: A $\mathbb{k}$-linear abelian category $\mathcal{C}$ is finite iff it is equivalent to a category "A-mod" of finite dim. modules over a finite dim. $\mathbb{k}$-algebra A.

Rk: Such A is not canonical (more precisely, it is unique just up to Morita equivalence)

... (Summary of last semester session 17) ...

Prop: A finite ring category $\mathcal{C}$ with left duals is a tensor category (i.e it is rigid, it has right duals)

Idea of the proof: show that the functor $(-)^*$ is essentially surjective,

i.e. $\forall X \in \mathcal{C}$, $\exists X' \in \mathcal{C}$ s.t. $(X')^* \cong X$

(it is enough to prove that because if $A^* \cong B$ then $A \cong {}^*B$)

the proof in details is in last semester session 17 (requiring 8 slides) (see also Prop 4.2.10 in the book).

Prop: Let P be a projective object in a multiring category. If $X \in \mathcal{C}$ has a left dual (resp. right) then the object $P \otimes X$ (resp. $X \otimes P$) is projective.

.... (summary of last semester session 18) ...

Prop: A locally finite abelian category is semisimple iff all objects are projective.

Corollary: If $\mathcal{C}$ is multiring category with left duals (e.g. multitensor) then $\mathbb{1} \in \mathcal{C}$ is projective iff $\mathcal{C}$ is semisimple

Rk: It extends Maschke's theorem (on finite groups)

[Let G be a finite group, $\Bbbk$ algebraically closed field.
Then: $\mathrm{Rep}_{\Bbbk}(G)$ is semi-simple iff $|G| \neq 0$ over $\Bbbk$]

because: $|G| \neq 0$ over $\Bbbk \iff$ trivial rep. $\Bbbk$ (i.e unit of $\mathrm{Rep}_{\Bbbk}(G)$) is projective

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Semisimplicity of the unit object:

Theorem: Let $\mathcal{C}$ be a multiring category. Then $\mathrm{End}_{\mathcal{C}}(1)$ is a semisimple algebra.

(the proof required 4 slides).

Rk: We already know that $\mathrm{End}_{\mathcal{C}}(1)$ is commutative, so by above, it must be a direct sum of finitely many copies of $\Bbbk$, so:

$$\mathrm{End}_{\mathcal{C}}(1) = \bigoplus_{i \in I} \Bbbk p_i, \text{ with } p_i \text{ primitive idempotent}$$

[$p \in R$ (ring) is called a primitive idempotent if $p^2 = p$ and pR indecomposable as R-module]

Let $1_i := \mathrm{Im}(p_i)$. Then: $1 = \bigoplus_{i \in I} 1_i$

Lemma:

- $\forall i$, 1_i is indecomposable
- $\forall i, j$, $1_i \simeq 1_j \iff i = j$
- $1_i \otimes 1_j \simeq 1_i \cdot \delta_{i,j}$ ← Kronecker symbol
- $1_i^* \simeq 1_i \simeq {}^* 1_i$

Corollary: In any multiring category $\mathcal{C}$, the unit object 1 is isomorphism to a direct sum of pairwise non-isomorphic indecomposable objects.

Let $\mathcal{C}_{ij} := 1_i \otimes \mathcal{C} \otimes 1_j$ $\left\{ \begin{array}{l} \text{objects: } 1_i \otimes X \otimes 1_j, \ X \in \mathcal{C} \\ \text{morphisms: } id_{1_i} \otimes f \otimes id_{1_j}, \ f \text{ morphism in } \mathcal{C} \end{array} \right\}$. Then:

(1) $\mathcal{C} = \bigoplus_{i,j \in I} \mathcal{C}_{ij}$, and $X \in \mathcal{C}$ indecomposable $\Rightarrow \exists i,j$ s.t. $X \in \mathcal{C}_{ij}$

(2) $\bigotimes_{\mathcal{C}} (\mathcal{C}_{ij} \times \mathcal{C}_{kl}) = \mathcal{C}_{il} \cdot \delta_{j,k}$ ($\triangle$ it is an abuse of notation).

(3) the category $\mathcal{C}_{ii}$ is a ring category ($\forall i$) with unit $\mathbb{1}_i$ } that justifies the term "multi"
(it is a tensor cat if $\mathcal{C}$ is multitensor)

(4) If $X \in \mathcal{C}_{ij}$ has left (or right) dual Y, then $Y \in \mathcal{C}_{ji}$

(the subcategories $(\mathcal{C}_{ij})$ are called component subcategories of $\mathcal{C}$)

Example: Let $\mathcal{D}$ be a ring cat. Then $\mathcal{C} = M_n(\mathcal{D})$ is a multiring cat, with $\mathcal{C}_{ij} \simeq \mathcal{D}$ as cat., $\mathcal{C}_{ii} \simeq \mathcal{D}$ as ring cat.

If $\mathcal{D} = Vec$, then $M_n(Vec)$ is a multitensor category.

See you next time ...