Lecture 8

6 April, 2022

Currents	Voltage
H_1 ~ 1-chain with finite energy	$\mathcal{D}$ ~ voltage with finite energy.
$\partial H_1 \subset H_0$ ~ 0-chain ~ external current	$f \in \ell_2 \Leftrightarrow \sum_x c(x) f^2(x) < \infty$

Fix $0 \in V$.

Def $\mathcal{D}$ space of functions $u : V \to \mathbb{R}$ s.t.

$$\|u\| := c(0)\,|u(0)| + D(u)^{\frac{1}{2}}$$

where

$$D(u) = \frac{1}{2} \sum_{x,y \in V} c(x,y)\left(u(y) - u(x)\right)^2$$

Notice: $D(u) = D(u+c)$ where c is constant function

Rmk: $\|u\| = 0 \iff u \equiv 0$.

We call D the Dirichlet space, with inner product

$$\langle u, v\rangle = c(0)\, u(0)\, v(0) + \frac{1}{2} \sum_{x \in Y} c(x,y) \big(u(y)-u(x)\big)\big(v(y)-v(x)\big)$$

Motivation:

(1) Effective resistance for infinite networks.

Idea: solve

$$(*) \begin{cases} \partial I + z = 0 \\ \langle I, Z \rangle = 0 \quad \text{for all finite cycles } Z. \end{cases}$$

and $z = \delta_a - \delta_b$

$$R_{eff}(a,b) = W(I) = u(a) - u(b)$$

Problem: (a) soln I is not unique generally.

(b) There are two special solns.

I_L limit current

I_M minimal current.

~~If I be any soln~~

To be done: "$W(I_M) \le \cancel{W(I) \le} W(I_k)$"

② We transform the problem

$$(*) \iff (**) \begin{cases} \partial I = 0 \\ \langle I - \bar{E}, z \rangle = 0 \quad \text{for all finite cycles } z \end{cases}$$

where $\bar{E}$ is a 1-chain s.t. $\partial \bar{E} = 2$.

(Check: Define $\tilde{I} := I - \bar{E}$,

$$\partial \tilde{I} = \partial I - \partial \bar{E} = 0 - 2 \iff \partial \tilde{I} + 2 = 0$$

$$\langle \tilde{I}, z \rangle = 0.$$
)

Z space of finite cycles.

$I \in Z \iff \partial I = 0$ and $I(xy) \neq 0$ for finitely many edges in Υ.

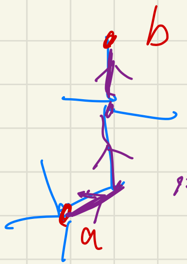
$$\partial I = 0$$
$$\sum_{y \in N(x)} I(x,y) = 0$$

Denote $\overline{Z}$ the closure of Z in H_1

Thm Let E be finite 1-chain. Then $\overline{Z}$ contains a unique soln to

$$\partial I = 0$$
$$\langle I - E, z \rangle = 0 \quad \text{for all finite cycles } z.$$

E.g. $z = \delta_a - \delta_b$



find E s.t. $\partial E = \delta_a - \delta_b$,

Take this E and E is finite.

pf: $Z < H_1$ is closed

$$\Rightarrow \quad H_1 = Z \oplus (Z)^{\perp}$$

$$E = I + K. \quad \leftarrow \text{decomposition is unique,}$$

Check: $(I \in Z) + (\partial : H_1 \to H_0 \text{ continuous}) \Rightarrow \; \partial I = 0$

$$\langle E, z \rangle = \langle I + K, z \rangle = \langle I, z \rangle$$

$\Leftrightarrow \quad \langle I - E, z \rangle = 0,$

$\Rightarrow \quad I$ is the unique soln in Z.

(i.e. orthogonal projection of E to Z.)

$\Rightarrow \quad \hat{I} := I - E$ solves $(*)$,

Prop: Suppose $E^{\dagger}$ is another finite 1-chain s.t. $\partial E^{\dagger} = z$ and $I^{\dagger}$ is soln of $(\cancel{**})$ with $E^{\dagger}$

Claim: $\hat{I} := I - E = I^{\dagger} - E^{\dagger}$

$\partial(E^{\dagger} - E) = z - z = 0$

Proof: Consider $J = \underbrace{(I - E)}_{\in Z} - \underbrace{(I^{\dagger} - E^{\dagger})}_{\in Z} = \underbrace{(I - I^{\dagger})}_{\in Z} + \underbrace{(E^{\dagger} - E)}_{\in Z}$

1 $\partial J = (\partial I - \partial \tilde{I}) - (\partial I^t - \partial \tilde{I}^t)$
$= (0 - z) - (0 - z) = 0.$

2. $\langle J, z \rangle = \langle I - \tilde{I}, z \rangle - \langle I^t - \tilde{I}^t, z \rangle = 0$ for all finite cycles z.

3. $J \in Z$. (true only if we consider finite 1-chains $\tilde{I}$).

2: $\Rightarrow$ $J \in Z^\perp \cap Z \Rightarrow J = 0$.

Def Suppose $z = \partial \Sigma$ for some finite 1-chain Σ,

Let I be the soln in $\overline{Z}$ s.t.

$$\partial I = 0$$

$$\langle I - \Sigma, Z \rangle = 0 \quad \text{for } Z \text{ finite cycles.}$$

Then $I_L := I - \Sigma$ is called the limit current,

It is independent of the choice of finit 1-chain Σ, generated by z.

Remark: If we consider $z = \partial \Sigma$ where Σ is not finite,

then $I - \Sigma \neq I_L$ generally.

Eg, $\lambda = 0$ $\Rightarrow$ $\mathcal{E} = 0$, $\Rightarrow$ $I_\lambda = 0$,

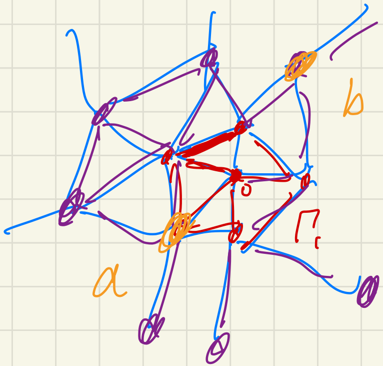
$$\partial I = 0 \quad \langle I, z \rangle = 0 \quad \Leftrightarrow \quad (Id - P)u = 0$$

$$W(I_\lambda) = 0 < D(u)$$

as long as u non-constant harmonic,

Rmk: I_λ "limit" current but not "largest".

Why I_L "limit" current?



$(\Gamma_1, r_1) \sim V_1 := \{x \in V \mid d(x,o) \leq 1\}$,

$(\Gamma_n, r_n) \sim V_n := \{x \in V \mid d(x,o) \leq n\}$,

$\Gamma_1 \subset \Gamma_2 \subset \dots \subset \Gamma_n \dots \subset \Gamma$

r_n the restriction of r to Γ_n.

Let ξ finite 1-chain s.t. $\mathrm{supp}(\xi) \subset \Gamma_n$ for some $n \in \mathbb{N}$.

Z_n finite cycles in Γ_n.

Consider over Γ_n, $\partial I_n = 0$

$$\langle I_n - E, z \rangle = 0 \quad \text{for all } z \text{ in } Z_n.$$

Extend I_n by zeros to Γ

$$I_n := \begin{cases} I_n(xy) & \text{if } xy \in \Gamma_n \\ 0 & \text{if } xy \notin \Gamma_n \end{cases}$$

$\Gamma_1 \subset \Gamma_2 \subset \dots \Gamma_n \subset \Gamma$ and $\bigcup_{n=1}^{\infty} \Gamma_n = \Gamma$.

Thm Fix finite 1-chain E. $\{\Gamma_n\}_{n \in \mathbb{Z}}$ exhaustion of Γ with finite subgraphs.

Then $\lim_{n \to \infty} \| I_L - I_n \|_{H_1} = 0$.

Pf: Let $\varepsilon > 0$. $\sim \partial J = 0$, J finite

$I_L \in Z \Rightarrow \exists\, J \in Z$ s.t. $\| I_L - J \| < \varepsilon$,

$\exists$ some n s.t. for $k > n$,

$\operatorname{Supp}(J) \subset \Gamma_n$.

For $z \in Z_K$, $\langle J - I_K, z \rangle = \langle J, z \rangle - \langle E, z \rangle$

take $z = J - I_K$ $\quad = \langle J, z \rangle - \langle I, z \rangle$

$= \langle J - I, z \rangle$

$\leq \| J - I \| \cdot \| z \|$

$< \varepsilon \| z \|$.

Note: $J - I_K \in Z_K \Rightarrow$

$$\|J - I_k\|^2 < \varepsilon \, \|J - I_k\|$$

$$\Rightarrow \quad \|J - I_k\| < \varepsilon,$$

$$\Rightarrow \quad \|I_L - I_k\| < \|I_L - J\| + \|J - I_k\| < 2\varepsilon$$

Since $\varepsilon > 0$ is arbitrary, $\Rightarrow \lim_{k \to \infty} \|I_L - I_k\|_{H_1} = 0$.

<u>Def.</u> $\quad Z = \delta_a - \delta_b \Rightarrow I_L$ minimal current

$\Rightarrow \quad R_L(a,b) := u(a) - u(b)$ is limit resistance.

Ex: $\{(\Gamma_n, r_n)\}$, $\rightsquigarrow$ I_n current in Γ_n generated by $\frac{2}{\delta_a - \delta_b}$.

Ex: $\{R_n\}$ is decreasing.

$\rightsquigarrow$ $R_n(a,b) = u_n(a) - u_n(b)$

$$\lim_{n\to\infty} R_n(a,b) = R_L(a,b) = W(I_L)$$

Def. Z^* space of all 1-cycles, infinite or finite

$$Z \subset Z^*$$

Check: Z^* is closed in H_1.

Let E be any ~~finite~~ 1-chain

$$\partial I = 0$$

$$\langle I - E, z \rangle = 0 \quad \text{for all } z \in Z^*$$

Find soln I in Z^*.

Ex: I exists and unique.

orthogonal projection of E into Z^*