$Sp(m) = U(2m) \cap Sp(2m, \mathbb{C})$

$Sp(m) \curvearrowright \mathbb{H}^m$. $\quad \psi: \mathbb{H}^m \times \mathbb{H}^m \to \mathbb{H}$.

$\psi = h + \varphi j$

$$\begin{cases} h: & \text{Hermitian form on } \mathbb{C}^{2m} \\ \varphi: & \text{symplectic form on } \mathbb{C}^{2m}. \end{cases}$$

$Sp(m) \subset U(2m) \curvearrowright \mathbb{C}^{2m}$

preserves $\varphi \Rightarrow$ preserves $\Lambda^m \varphi$

$\varphi = dz_1 \wedge dz_2 + dz_3 \wedge dz_4 + \cdots + dz_{2m-1} \wedge dz_{2m}$

then $\quad \Lambda^m \varphi = \underbrace{*}_{\neq 0} \cdot dz_1 \wedge \cdots \wedge dz_{2m}$

$\Rightarrow \quad Sp(m)$ preserves $dz_1 \wedge \cdots \wedge dz_{2m} \in \Lambda^{2m} \mathbb{C}^*$.

$\Rightarrow \quad Sp(m) \subset SU(2m)$.

(M, g) Rie. mfd, $\dim = n$.

- If $n = 2m$, $\quad Hol = SU(m) < SO(2m)$,

  then by prallel transport $\rightsquigarrow J, \omega$

  (M, J, g) Kähler mfd, $\quad \omega$ holomorphic m-form nowhere vanishing.

(M, J, g, ω) is a Calabi-Yau mfd together with a Ricci-flat metric.

· If $n=4m$, $\mathrm{Hol} = Sp(m) \subset SO(4m)$,

I, J, K, φ on M.

$\forall\, a, b, c \in \mathbb{R}$ s.t. $a^2+b^2+c^2=1$,

$(M, aI+bJ+cK, g)$ Kähler mfd.

$$(aI+bJ+cK)^2 = -a^2-b^2-c^2+ab(IJ+JI)+bc(JK+KJ)$$
$$+ac(IK+KI)$$
$$=-1$$

Many Kähler structures on M.

This is the reason of the name "hyper-Kähler".

Precisely, we define a hyper-Kähler mfd to be

a simply-connected Riemannian mfd (M, g) with

$\mathrm{Hol} = Sp(m) \subset SO(4m)$ (thus we have I, J, K on M)

(M, I, J, K, g) hyper-Kähler, $\dim_{\mathbb{C}} M = 2m$.

φ: $\underbrace{\text{non deg.}}_{\text{means}}$ holo. 2-form on M

$\wedge^m \varphi$ is a nowhere vanishing holo. 2m-form.

Defn: (M, J) is called a holomorphic symplectic manifold, if

it is a complex manifold with a everywhere non deg. holo. 2-form.

(M, I, J, K, g)

So, a hyper-Kähler mfd, considered as a complex manifold $(M, aI + bJ + cK)$ for any $(a,b,c) \in S^2$, is a holomorphic symplectic manifold

Conversely, given a holo. symplectic manifold (M, J) can we always find a Riemann metric g, such that (M, J, g) is hyper-Kähler.

Need to add conditions.

Fact: A compact simply-connected holo. symplectic mfd is not necessarily Kählerian

D. Guan 1994: An example: Simply-conn. compact holo. symplecti mfd that is not Kählerian.

(disprove a previous conj. by Todorov)

As a contrast, Siu: complex K3 surface is always Kählerian.

So we need to consider Kählerian complex mfd.

Calabi Conj (1954 by Calabi, proved by Yau 1976)

(M, J) compact complex mfd, $\exists\, g$ a Kähler metric on M, with ω the Kähler form, Suppose ρ' is a real, closed $(1,1)$-form on M with

$[\rho'] = 2\pi \, c_1(M)$ $= 0$ when K_M trivial, can take $\rho' = 0$.

Then $\exists!$ Kähler metric g' with Kähler form ω'.

such that $[\omega'] = [\omega] \in H^2(M, \mathbb{R})$, and the Ricci form

of g' is ρ' $= 0$, means g' is Ricci-flat.

$\dim_{\mathbb{C}} M = n$.

(M, J) is a Calabi-Yau mfd (automatically Kählerian)

then by Yau thm, $\exists$ Rie. metric g on M, such

that g is Ricci-flat,

$\Rightarrow$ the nowhere vanishing holo. n-form on M is a

constant tensor w.r.t. g

$\Rightarrow$ $\mathrm{Hol} \subset SU(n)$.

Need more conditions to make $\mathrm{Hol} = SU(n)$.

If (M, J) is Kählerian compact holo. symplectic manifold, $\dim_{\mathbb{C}} M = 2m$.

φ: holo. 2-form $\qquad \Lambda^m \varphi$: nowhere vanishing holo. $2m$-form.

Yau thm: $\exists$ Ricci-flat Riemann metric g,

[Principle: on Ricci-flat Kähler mfd, every holo. tensor is constant].

So φ is a constant tensor.

$x \in M$, $\underbrace{T_x M.\ J_x\ \ \varphi_x}$

Hol_x fixes $\underbrace{J_x, g_x}$, $\underbrace{\varphi_x}$.

$\mathrm{Hol}_x \subset U(2m) \cap Sp(2m, \mathbb{C}) = Sp(m)$.

Need more conditions to make $\mathrm{Hol} = Sp(m)$.

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to construct HK, need to first construct Holo. symplectic,

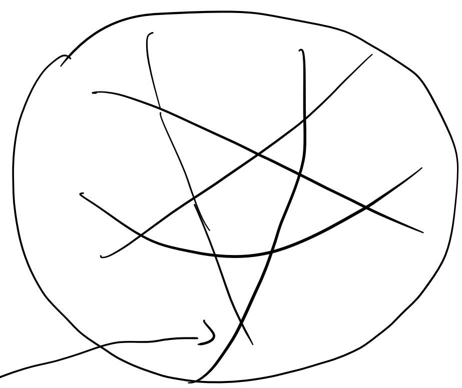
then apply Yau thm.

- $K3^{[n]}$: Hilbert scheme of pts on K3.
- $K^{[n]}(A)$: generalized Kummer.
- O'Grady 6 & O'Grady 10.

They are complex mfds at beginning. can be shown to be holo. symp. mflds.

$K3^{[n]}$. type

n fixed.



$\dim_{\mathbb{C}}(\text{Moduli}) = 20$.

non-Hausdorff analytic space.

dense. each comp has dim 19.

Contains algebraic holo. sympl. mfd.

They are Kählerian by existence of the Fubini-Study metric. $\underset{\text{Yau thm}}{\Longrightarrow}$ $\exists$ Ricci-flat metric.

Fubini - Study metric

$$\mathbb{C}P^n = (\mathbb{C}^{n+1} - 0)/\mathbb{C}^\times = \left\{ [Z_1 : \cdots : Z_{n+1}] \;\middle|\; \begin{array}{l} Z_1, \ldots, Z_{n+1} \in \mathbb{C} \\ \text{not all zero} \end{array} \right\}.$$

is a complex manifold

$i \partial \bar\partial \log(|Z_1|^2 + \cdots + |Z_{n+1}|^2)$ real closed $(1,1)$-form on $\mathbb{C}^{n+1} - 0$.

pulled back to $\mathbb{C}P^n$ via

holo. sections of $\mathbb{C}^{n+1} - 0 \longrightarrow \mathbb{C}P^n$.

<u>Check</u>: it is globally well defined on $\mathbb{C}P^n$.

because for holo. function $f : U \to \mathbb{C}$. $\quad U \subset \mathbb{C}P^n$

we have $i \partial \bar\partial (|f|^2) = i \partial \bar\partial (f \cdot \bar f) = 0$

Want to show this is a Kähler form.

Affine chart $\{Z_0 \neq 0\} \subset \mathbb{C}P^n$.

$$\begin{array}{ccc} Z_0 \cdots Z_n & & \\ \downarrow & & \cong \\ z_i = Z_i / Z_0 & & \mathbb{C}^n. \end{array}$$

$$i \partial \bar\partial \log(1 + |z_1|^2 + \cdots + |z_n|^2) \qquad z = (z_1, \cdots, z_n)$$

$$= i \partial \left[\frac{z \cdot d\bar z}{1 + |z|^2} \right]$$

$$= i \frac{(1 + |z|^2)\, dz\, d\bar z - \partial(|z|^2) \cdot (z \cdot d\bar z)}{(1 + |z|^2)^2}$$

$$= i \cdot \frac{(1+|z|^2)\, dz\, d\bar{z} - (\bar{z}\cdot dz)\cdot(z\cdot d\bar{z})}{(1+|z|^2)^2}$$

Cauchy inequality $\Rightarrow$ the asso. symmetric form is pos. def.

$\mathbb{C}P^n$ has a natural Kähler str.

A quasi-proj. complex smooth variety is a complex submanifold of $\mathbb{C}P^n$, hence inherit a Kähler metric by taking restriction of the Fubini-Study metric

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Polynomials $\rightsquigarrow$ alg. var. $\xrightarrow{\text{Fubini-Study}}$ Kähler mfd.

Yau. thm

mfd with good metric.

Can already calculate canonical bundle, simply-conn.
