$\mathbb{F}$ field, $Sp(\mathbb{F}, 2m) := \left\{ A \in GL(\mathbb{F}, 2m) \;\middle|\; A\begin{pmatrix} & I \\ -I & \end{pmatrix}A^T = \begin{pmatrix} & I \\ -I & \end{pmatrix} \right\}$

$Sp(m) = \left\{ A \in U(2m) \;\middle|\; A\begin{pmatrix} & I \\ -I & \end{pmatrix}A^T = \begin{pmatrix} & I \\ -I & \end{pmatrix} \right\}$,

$\shortparallel$

$Sp_{\mathbb{C}}(m)$ also called unitary symplectic group.

An error last time: I defined $Sp(m)$ to be

$$\left\{ A \in U(2m) \;\middle|\; A\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} A \right\}$$

The correct one: $A\begin{pmatrix} & I \\ -I & \end{pmatrix} = \begin{pmatrix} & I \\ -I & \end{pmatrix}\overline{A}$

$\mathbb{H} = \mathbb{R}\{1, i, j, k\}$.

Let $\mathbb{C} = \mathbb{R}\{1, i\}$, then $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j \xrightarrow{\cong} \mathbb{C}^2$

as complex vector spaces.

$$\mathbb{H}^m \xrightarrow{\cong} \mathbb{C}^{2m}.$$

$$u + vj \longmapsto (u, v).$$

for $u, v \in \mathbb{C}^m$.

(row vectors)

isomorphism as $\mathbb{C}$-vector spaces, while the scalar is from the left side

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Define $GL_{\mathbb{H}}(\mathbb{H}^m)$ to be the group of $\mathbb{H}$-linear automorphisms of $\mathbb{H}^m$.

Precisely,

$g \in GL_{\mathbb{H}}(\mathbb{H}^m)$, if

$g: \mathbb{H}^m \longrightarrow \mathbb{H}^m$ map s.t. $g(\lambda v) = \lambda g(v)$, $\forall \lambda \in \mathbb{H}, v \in \mathbb{H}^m$

$e_1, e_2, \cdots, e_m$ standard $\mathbb{H}$-basis of $\mathbb{H}^m$.

$$\begin{pmatrix} g e_1 \\ g e_2 \\ \vdots \\ g e_m \end{pmatrix} \in M(m \times m, \mathbb{H}).$$

$$=: A + Bj, \quad \text{where } A, B \in M(m \times m, \mathbb{C})$$

Then for any $u + vj \in \mathbb{H}^m$,

write $u + vj = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_m e_m$.

$$\begin{aligned} g(u+vj) &= g(\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_m e_m) \\ &= \lambda_1 g(e_1) + \lambda_2 g(e_2) + \cdots + \lambda_m g(e_m) \\ &= (\lambda_1, \cdots, \lambda_m) \begin{pmatrix} g(e_1) \\ \vdots \\ g(e_m) \end{pmatrix} \\ &= (u+vj)(A+Bj) \\ &= uA + \boxed{vj} Bj + uBj + vjA \\ &= (uA - v\bar{B}) + (uB + v\bar{A})j \end{aligned}$$

$$g: \mathbb{H}^m \longrightarrow \mathbb{H}^m.$$
$$u+vj \longmapsto (uA - v\bar{B}) + (uB + v\bar{A})j$$

when $\mathbb{H}^m$ is regarded as $\mathbb{C}^{2m}$.

$$g: \mathbb{C}^{2m} \longrightarrow \mathbb{C}^{2m}.$$

$(A \quad B)$

$$(u,v) \longmapsto (uA - v\bar{B},\ uB + v\bar{A}) = (u,v)\begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix},$$

We obtain a group embedding:

$$GL_{\mathbb{H}}(\mathbb{H}^m) \hookrightarrow GL_{\mathbb{C}}(\mathbb{C}^{2m})$$

$$A + Bj \longmapsto \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}$$

Take $X \in GL_{\mathbb{C}}(\mathbb{C}^{2m})$,

then X is the image of an element in $GL_{\mathbb{H}}(\mathbb{H}^m)$

if and only if X commutes with the left

multiplication of j on $\mathbb{H}^m = \mathbb{C}^m \oplus \mathbb{C}^m j$

take $u, v \in \mathbb{C}^{2m}$, $u + vj \in \mathbb{H}^m$.

$$j(u+vj) = ju + jvj = \bar{u}j + j^2\bar{v} = -\bar{v} + \bar{u}j$$

So on $\mathbb{C}^{2m}$, the left multiplication of j sends

$$(u,v) \text{ to } (-\bar{v}, \bar{u}) = (\bar{u}, \bar{v})\begin{pmatrix} & I \\ -I & \end{pmatrix}$$

$$j\,[(u,v)X] = [j(u,v)]\,X$$

$$\overline{(u,v)X}\begin{pmatrix} & I \\ -1 & \end{pmatrix} \qquad (\bar{u},\bar{v})\begin{pmatrix} & I \\ -I & \end{pmatrix} X$$

$$(\bar{u},\bar{v})\bar{X}\begin{pmatrix} & I \\ -1 & \end{pmatrix} \quad // \quad \Rightarrow \bar{X}\begin{pmatrix} & I \\ -I & \end{pmatrix} = \begin{pmatrix} & I \\ -I & \end{pmatrix} X.$$

$\Updownarrow$

X is of the form $\begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}$

Conclusion: $GL_{\mathbb{H}}(\mathbb{H}^m) \hookrightarrow GL_{\mathbb{C}}(\mathbb{C}^{2m})$

$$A + Bj \longmapsto X = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}$$

$$\gamma : \mathbb{H}^m \times \mathbb{H}^m \longrightarrow \mathbb{H}$$

$$z,\ z' \longmapsto \sum z_\ell \overline{z'_\ell}$$

Look at the subgroup of $GL_{\mathbb{H}}(\mathbb{H}^m)$ preserving ψ

Write out the expression of ψ as a function on $\mathbb{C}^{2m} \times \mathbb{C}^{2m}$

Take $u + vj,\ u' + v'j \in \mathbb{H}^m = \mathbb{C}^m \oplus \mathbb{C}^m j$

$$\psi(u+vj,\ u'+v'j)$$

$$= \sum_{1 \le \ell \le m} (u+vj)_\ell \, \overline{(u'+v'j)}$$

$\overline{v'j} = -v'j$

Since $v'j = *j + *k$

$$= \sum (u_\ell + v_\ell j)(\overline{u'_\ell} - v'_\ell j)$$

$$= \sum \left(u_\ell \overline{u'_\ell} + v_\ell j(-v'_\ell)j\right) + \sum \left(v_\ell j \overline{u'_\ell} - u_\ell v'_\ell j\right)$$

$$= \sum \left(u_\ell \overline{u'_\ell} + v_\ell \overline{v'_\ell}\right) + \sum \left(v_\ell u'_\ell - u_\ell v'_\ell\right) j$$

Define $h, \varphi : \mathbb{C}^{2m} \times \mathbb{C}^{2m} \longrightarrow \mathbb{C}$

$$h((u,v),(u',v')) = \sum \left(u_\ell \overline{u'_\ell} + v_\ell \overline{v'_\ell}\right)$$

$$\varphi((u,v),(u',v')) = \Sigma (v_l u'_l - u_l v'_l)$$

h is the standard Hermitian inner product on $\mathbb{C}^{2n}$.

φ is the standard symplectic form on $\mathbb{C}^{2m}$.

Conclusion: $\psi = h + \varphi j$.

take $g \in GL_{\mathbb{H}}(\mathbb{H}^m) \longleftrightarrow X \in GL_{\mathbb{C}}(\mathbb{C}^{2m})$.

If $g \curvearrowright \mathbb{H}^m$ preserves ψ

$\Rightarrow g \curvearrowright \mathbb{C}^{2m}$ preserves h, φ

g preserves $h \iff X \in U(2m)$

g preserves $\varphi \iff g \in Sp(2m, \mathbb{C})$

$$\varphi((u,v),(u',v')) = \sum (v_l u'_l - u_l v'_l)$$

$$= (u,v) \begin{pmatrix} & -I \\ I & \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}$$

g preserves $\varphi \iff \varphi[(u,v),(u',v')] = \varphi((u,v)X, (u',v')X)$

$$\iff (u,v) \begin{pmatrix} & -I \\ I & \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = (u,v) X \begin{pmatrix} & -I \\ I & \end{pmatrix} X^t \begin{pmatrix} u' \\ v' \end{pmatrix}$$

$$\iff X \begin{pmatrix} & -I \\ I & \end{pmatrix} X^t = \begin{pmatrix} & -I \\ I & \end{pmatrix}.$$

Summary: for $X \in GL_{\mathbb{C}}(\mathbb{H}^m)$,

① X is $\mathbb{H}$ linear $\iff$ $X \begin{pmatrix} & I \\ -I & \end{pmatrix} = \begin{pmatrix} & I \\ -I & \end{pmatrix} \bar{X}$

$\iff$ X is of the form $\begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}$

② X preserves h $\iff$ $X\bar{X}^t = I$

$\iff$ $X \in U(2m)$

③ X preserves φ $\iff$ $X \begin{pmatrix} & I \\ -I & \end{pmatrix} X^t = \begin{pmatrix} & I \\ -I & \end{pmatrix}$

observation: Assuming ②, then ① $\iff$ ③

① + ② $\iff$ ② + ③ $\Rightarrow$ $X \in U(2m) \cap Sp(2m, \mathbb{C}) = Sp(m)$

$Sp(m) \ni X = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}$.

$|\det X| = 1$.

Actually $\det X = 1$. i.e. $Sp(m) < SU(2m)$.

enough to show $\det \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \in \mathbb{R}^{\geq 0}$.

can assume wlog A invertible.

$$\begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \begin{pmatrix} I & -A^{-1}B \\ & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ -\bar{B} & \bar{B}A^{-1}B + \bar{A} \end{pmatrix}$$

$$\det \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} = \det(A) \cdot \det(\bar{B}A^{-1}B + \bar{A})$$

$$= \underbrace{\det(A)\cdot \det(\bar{A})}_{>0} \cdot \det(\bar{A}^{-1}\bar{B}A^{-1}B + I)$$

Denote $C = A^{-1}B$,

$$\det(I + \bar{C}C) \overset{?}{>} 0$$

(M, J) complex manifold,

a Riemann metric g on M is called Hermitian if

$$g(Jv, Jw) = g(v, w)$$

$\omega(v, w) = g(Jv, w)$ real $(1,1)$-form.

(M, J, g) is called a Kähler manifold if

$$\begin{cases} d\omega = 0 \\ \nabla\omega = 0 \\ \nabla J = 0 \end{cases}$$ equivalent conditions.

If (M, J), as a complex manifold, admits Kähler metric,

then we call (M, J) is Kählerian.

For example: Complex K3 surfaces are Kählerian.

(Siu)

$\dim_{\mathbb{C}} M = n$.

(M, J, g) Kähler,

$\mathrm{Hol} \curvearrowright T_xM$, fix J_x, g_x.

$\Rightarrow$ $\mathrm{Hol} \subset U(n)$.

Conversely, assume a Rie. Mfd (M, g) such that:

$$\mathrm{Hol} \subset U(n) \subset SO(T_xM)$$

then we can choose $J_x : T_xM \to T_xM$ complex structure,

such that $\begin{cases} g_x(J_x v, J_x w) = g_x(v, w) \quad \forall v, w \in T_xM \\ \mathrm{Hol} \text{ fixes } J_x. \end{cases}$

By parallel transport, J_x can be extended to an

almost complex str. on M, compatible with g.

(constant tensor)

By theory of 多复变: J is integrable.

(M, J, g) is a Kähler manifold.

Fact: All complex smooth alg. varieties are Kählerian

complex manifolds.

For compact complex manifolds that are Kählerian, one can

construct so-called Hodge decomposition on $H^*(\cdot; \mathbb{C})$

this does not depend on the choices of the metric.

$SU(n) \longleftrightarrow$ Calabi-Yau $\qquad Sp(m) \longleftrightarrow$ hyper-Kähler mfds.

If (M,g) is a Riemannian Mfd such that:

$$\mathrm{Hol} \subset SU(n) < SO(T_xM).$$

then there exists $J_x : T_xM \to T_xM$. $J_x^2 = -1$

compatible with g_x.

$\exists \ \omega_x \in \Lambda^n T_x^* M$.

such that Hol fixes J_x, ω_x.

by parallel transport, J_x, ω_x can be extended to

J, ω on M.

(M, J, g) is a Kähler mfd,

ω is a $\underbrace{(n,0)\text{-form}}_{\text{locally } * dz_1 \wedge \cdots \wedge dz_n}$ such that $\nabla \omega = 0$.

$\Downarrow$ fact.

canonical line bundle

$\bar{\partial} \omega = 0$, i.e. ω holomorphic

So K_M admits a holomorphic section, everywhere nonzero
ω

$\Rightarrow$ K_M is a trivial line bundle $\Rightarrow$ (M,J,g) Calabi-Yau mfd.

Actually: the "curvature" of K_M vanishes.

$\shortparallel$

Ricci curvature.

$\Rightarrow$ (M, J, g) is Calabi-Yau such that g is Ricci-flat

If (M, g) is a Riemannian mfld such that:

$$\mathrm{Hol} \subset Sp(m) < SU(2m) \subset SO(T_xM).$$

Can realize T_xM as an $\mathbb{H}$-vector space.

$$T_xM \cong \mathbb{H}^m.$$

$$\Psi : \mathbb{H}^m \times \mathbb{H}^m \longrightarrow \mathbb{H}$$

$\Psi_x = h_x + \varphi_x j_x$, $\quad h_x, \varphi_x : \mathbb{C}^{2m} \times \mathbb{C}^{2m} \longrightarrow \mathbb{C}$

extends (by parallel transport) h_x, Ψ_x to h, φ over M
i_x, j_x, k_x $\quad i, j, k$

then $(M, ai+bj+ck, g)$
is a Kähler mfd, $\forall (a, b, c) \in S^2$.

$\mathbb{P}^1 \longleftrightarrow S^2$-family of complex str. on (M, g)

[twistor family.].

$\varphi_x \rightsquigarrow \varphi$: $(2,0)$ type form on M,
constant tensor.

Fact: φ is holomorphic.

φ is a holo. nondeg. 2-form on M.

$\Longrightarrow (M, ai+bj+ck, g, \varphi)$ is a hyper-Kähler mfd