Lecture 6 30 March

Next lecture: 2 April (Sat) 15:20-16:50

No lecture on 4 April.

Green's function with an absorbing vertex $b \in V$.

$$p(x,y;b) := \begin{cases} p(x,y) & \text{if } x \neq b,\ y \neq b, \\ 0 & \text{if } x = b,\ y \neq b, \\ 1 & \text{if } x = b,\ y = b. \\ p(x,y) & \text{if } x \neq b,\ y = b. \end{cases}$$

For every x,

$$G(x,y;b) = \begin{cases} \sum_{n=0}^{\infty} p^n(x,y;b) & \text{if } y \neq b \\ 0 & \text{if } y = b, \end{cases}$$

$G(x, y; b)$ is expected number of visits from x to y before hitting b in the case $y \neq b$.

When $y \neq b$,

$$G(b, y; b) = 0$$

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Ex: $G(x, y; b)$ is finite even if (Γ, r) is recurrent.

Claim: $u := G(f)$ solves the equations $(*)$

i.e. $$u(x) = \sum_{\substack{y \in V \\ y \neq b}} G(x,y;b)\, f(y)$$

Check: 1 $u(b) = 0$?

Compute: $$u(b) = \sum_{y \in V} G(b,y;b)\, f(y) = 0$$

not $p(x,y;b)$

2 $(Id - P)\,u = f$ for $x \in V - \{b\}$

Consider $$\sum_{y \in V} p(x,y)\, u(y) = \sum_{\substack{y \in V \\ y \neq b}} p(x,y)\, u(y)$$

(Since $x \neq b$, $y \neq b$,)

$$= \sum_{\substack{y \in V \\ y \neq b}} p(x,y;b)\, u(y)$$

$$= \sum_{\substack{y \in V \\ y \neq b}} p(x,y;b) \sum_{\substack{z \in V \\ z \neq b}} G(y,z;b) f(z)$$

$$= \sum_{z \neq b} \sum_{n=1}^{\infty} p^n(x,z;b) \quad f(z)$$

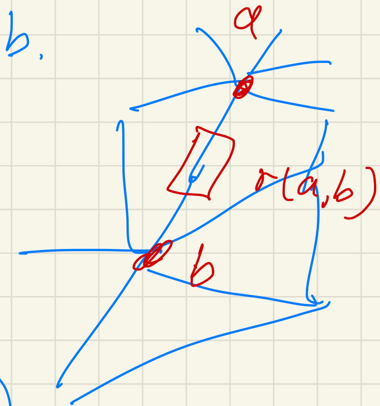
$$= \sum_{z \neq b} \Big(G(x,z;b) - Id(x,z) \Big) f(z)$$

$$= u - f(x)$$

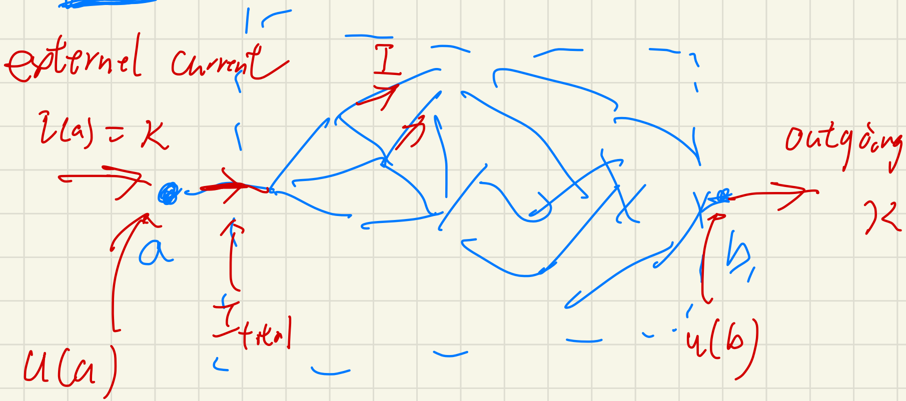
$$\Rightarrow \quad (Id - P)u(x) = f(x) \qquad \text{for } x \in V - \{b\}.$$

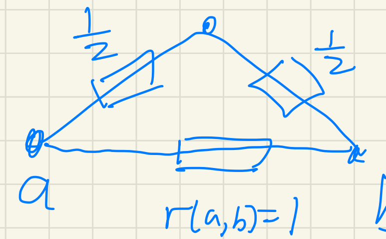
Effective resistance R between a and b.

If $a \sim b$, $\quad r(a,b) \geq R(a,b)$.



Def





$R(a,b) = \frac{1}{2}$

$$\Delta V = I R$$

$$R_{eff}(a,b) := R = \frac{u(a)-u(b)}{I_{total}(a,b)} = \frac{u(a)-u(b)}{K}$$

where u is potential of current I such that $\partial I = K(\delta_a - \delta_b)$

← total current from a to b.

We can plug in $u = G(f)$.

Recall :

$$\sum_{y \in V(x)} c(x,y)\big(u(y)-u(x)\big) + l(x) = 0$$

$$\Rightarrow \quad \sum \frac{c(x,y)}{c(x)} \big(u(y)-u(x)\big) = -\frac{l(x)}{c(x)}$$

"$(Id-P)u = f$" $\qquad$ "$(P-Id)u(x)$"

$$\Rightarrow \quad (Id-P)\,u(x) = \frac{l(x)}{c(x)} = f(x).$$

Take $f(a) = \dfrac{K}{c(a)}$, $f(b) = \dfrac{-K}{c(b)}$

$$\boxed{f(x) = \begin{cases} \dfrac{K}{c(a)} & x = a \\ -\dfrac{K}{c(b)} & x = b \\ 0 & \text{otherwise} \end{cases}}$$

Make vertex v as absorbing.

$$R(a,b) = \frac{u(a) - u(b)}{K}$$

$$= \frac{1}{K}\left(\sum_y G(a,y;v) f(y) - \sum_y G(b,y;v) f(y) \right)$$

$$= \frac{1}{K}\left(\left(G(a,a;v)\frac{K}{c(a)} - \frac{K}{c(b)} G(a,b;v) \right) - \left(G(b,a;v)\frac{K}{c(a)} - \frac{K}{c(b)} G(b,b;v) \right) \right)$$

$$(*) = \frac{G(a,a;v) - G(b,a;v)}{c(a)} + \frac{G(b,b;v) - G(a,b;v)}{c(b)}$$

In case choose $v = b$, then know: $u(b) = 0$

$(*)$ becomes $\Rightarrow R(a,b) = \dfrac{G(a,a;b)}{c(a)} > 0$

Thm (Foster's averaging formula for finite graphs).

$n :=$ number of vertices $< \infty$.

sum over all edges

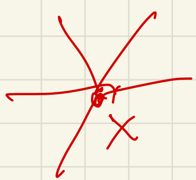
$$\sum_{x \sim y} \frac{R(x,y)}{r(x,y)} = 2(n-1).$$

Pf: $$\frac{R(x,y)}{r(x,y)} = \frac{G(x,x;V) - G(y,x;V)}{c(x)\, r(x,y)} + \frac{G(y,y;V) - G(x,y;V)}{c(y)\, r(x,y)}$$

Recall: $$\boxed{\frac{1}{c(x)\, r(x,y)} = \frac{c(x,y)}{c(x)} = p(x,y)}$$

$$= p(x,y)\big(G(x,x;V) - G(y,x;V)\big) + p(y,x)\big(G(y,y;V) - G(x,y;V)\big)$$

$$\sum_{x\sim y} \frac{R(x,y)}{r(x,y)} \overset{?}{=} 2 \sum_{x\notin V} \sum_{y\in V(x)} p(x,y)\Big(G(x,x;V) - G(y,x;V)\Big)$$



$$= 2 \sum_{x\notin V} \Big(G(x,x;V) - \sum_{y\in V(x)} p(x,y)\, G(y,x;V) \Big)$$

$$= 2 \sum_{x\notin V} \Big(\big[(Id - P)\, G(\cdot,x;V)\big](x) \Big)$$

$$= 2 \sum_{x \neq v} \operatorname{Id}(x, x)$$

$$= 2(n-1)$$

Lemma:

0-cochain u, 1-chain z

1-cochain $\partial^* u$ 0-chain ∂z

$$(u, \partial z) = (\partial^* u, z)$$

$$\parallel \qquad\qquad \parallel$$

Ex:

$$\sum_{x \in V} u(x) \sum_{y \in V(x)} z(x,y) = \sum_{[x,y] \in \vec{E}} (u(y) - u(x))\, z(x,y)$$

Def. Let η be 1-chain. We define

$$W(\eta) = \frac{1}{2} \sum_{[x,y] \in Y} r(x,y) \left(\eta(x,y) \right)^2$$

$[x,y] \in X \iff [y,x] \in X \leftarrow X$ collection of oriented edges

$[x,y] \in Y \Rightarrow [y,x] \notin Y \leftarrow Y$ collection of unoriented edges.

Thm (Thomson's principle).

i 0-chain represents external current.

$\Rightarrow$ I physical current that solves Poisson equation.

Let $\tilde{I}$ be any other 1-chain such $\partial \tilde{I} = i$

($\tilde{I}$ satisfy $\checkmark$ $\partial\tilde{I} = i$
$\times$ $\langle R^{+}\tilde{I}, z\rangle = 0$ for all cycles z)

Then $W(\tilde{I}) \geq W(I)$

and equality holds if and only if $I = \tilde{I}$.

Pf: Let u be the potential of I.

$$I(x,y) = \frac{u(y)-u(x)}{r(x,y)},$$

Let $D = \tilde{I} - I \Rightarrow \partial D = \partial \tilde{I} - \partial I$

$$= 2 - 2 = 0.$$

$$\sum r(x,y)\, I(x,y)\, D(x,y) = \sum (u(y)-u(x))\, D(x,y)$$

$$= \sum_{x} u(x) \sum_{y \in V(x)} D(x,y)$$

$$= 0$$

$$W(\tilde{I}) = \frac{1}{2} \sum r(x,y) \left(I(x,y) + D(x,y) \right)^2$$

$$(\tilde{I} = I + D) \quad = \frac{1}{2} \sum r(x,y) \left(I(x,y) \right)^2 + \frac{1}{2} \sum r(x,y) I(x,y) D(x,y)$$

$$+ \frac{1}{2} \sum r(x,y) \left(D(x,y) \right)^2$$

$$= W(I) + 0 + W(D)$$

$$\geq W(I) \left(= R_{eff}(a,b) \text{ if } \partial I = \delta_a - \delta_b \right).$$

Equality holds $\iff W(D) = 0 \iff D \equiv 0.$

Physics:

$V = IR \rightsquigarrow R := \frac{V}{I}$

Energy: $E = I^2 R = \frac{V^2}{R} = IV$

flow of charges

energy carried by each charge.

If $I = 1$, $E = R_{eff}$.