Recall: $f: X \to Y$ étale covering, we say that it's Nisnevich if $\forall y \in Y, \exists x \in f^{-1}(y)$ s.t. $k(x) = k(y)$. We could define Nis sheaves with transfers F to be those presheaves satisfying Nis sheaf condition.

$$F \text{ is a presheaf on } X,\ x \in X,\ F_x = \varinjlim_{\substack{U \xrightarrow{ét} X \\ u \mapsto x \\ k(u)=k(x)}} F(U) = F(\mathcal{O}^h_{X,x}).$$

Prop 2.22 Let $0 \to F \to G \to H \to 0$ be a complex in $Sh(\mathcal{C}_X)$, TFAE

1) The complex is exact
2) For every $x \in X$, the complex of abelian groups
$$0 \to F_x \to G_x \to H_x \to 0$$
is exact.

Proof: Suppose $F \in PSh(\mathcal{C}_X)$, define its sheafification F^+ as the following

$u \mapsto x$, $\begin{pmatrix} U \\ \downarrow ét \\ X \end{pmatrix}$, $k(u)=k(x)$

$$F^+(U) = \left\{ (u_x) \in \prod_{x \in U} F_x \;\middle|\; \forall x \in U\ \exists\ V \xrightarrow[f]{Nis} U,\ s \in F(V),\ \forall \underset{\in V}{y} \text{ if } k(y) = k(f(y)) \Rightarrow s_y = u_{f(y)} \right\}$$

$U \xrightarrow{ét} X$, $u \mapsto x$, $k(u)=k(x)$, $F_u = F_x$

$x \in U \to X$, $V \to U$, $f: V \to X$, $k(v) = k(f(v)),\ \forall v \in V$

When $F^+ = 0$? If $\forall s \in F(U)$, $\underset{x \in}{V} \to U$ $s|_V = 0$ $\Leftrightarrow F^+ = 0$

1) $\Rightarrow$ 2) If the complex is exact, then $0 \to F_x \to G_x \to H_x \to 0$ exact because the $\varinjlim$ is an exact functor.

2) $\Rightarrow$ 1) $0 \to F \to G \xrightarrow{\pi} H \to 0$ $\square$

$$\operatorname{coker}(\pi) = (H/G)^+ = 0$$

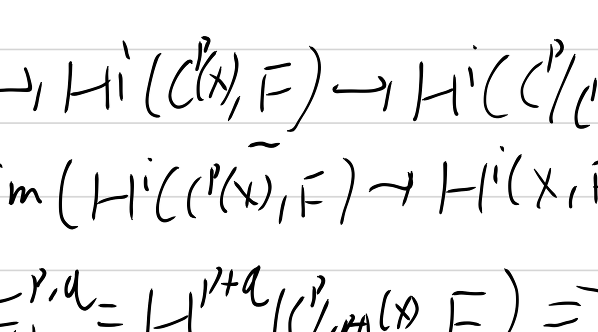
For any X noetherian with $\dim X < \infty$, define $C^p(X) = \bigoplus_{\substack{y \in X \\ \operatorname{codim} \bar{y} \geq p}} \mathbb{Z} \cdot \bar{y}$. Fix $F \in Sh(\mathcal{C}_X)$, $Z \subseteq W \subseteq X$ (closed), $Z \in C^{p+1}(X)$, $W \in C^p(X)$. We have a long exact sequence

$$\cdots \to H^i_Z(X,F) \to H^i_W(X,F) \to H^i_{W \setminus Z}(X \setminus Z, F) \to H^{i+1}_Z(X,F) \to \cdots$$

Define $0 \to F_Z(X) \to F(X) \to F(X \setminus Z)$

$H^i_Z(X,F)$ = i-th right derived functor

Define $\underline{H^i(C^p(X), F)} = \varinjlim_{Z \in C^p(X)} H^i_Z(X,F)$, $H^i(C^p/C^{p+1}(X), F) = \varinjlim_{\substack{Z \subseteq W \\ W \in C^p,\ Z \in C^{p+1}}} H^i_{W\setminus Z}(X \setminus Z, F)$

$W \subseteq X$  $W \setminus Z \subseteq X \setminus Z$

Taking limit with respect to pairs $Z \subseteq W$, $W \in C^p(X)$, $Z \in C^{p+1}(X)$, we obtain an exact sequence

$$\to H^i(C^{p+1}(X), F) \to H^i(C^p(X), F) \to H^i(C^p/C^{p+1}(X), F) \to H^{i+1}(C^{p+1}(X), F) \to$$

So set $F^p H^i(X,F) = \operatorname{Im}(H^i(C^p(X), F) \to H^i(X,F))$, we obtain a spectral sequence

$$E_1^{p,q} = \underline{H^{p+q}(C^p/C^{p+1}(X), F)} \Rightarrow H^{p+q}(X,F),$$

called the <u>coniveau</u> spectral sequence. $E_1^{p,q} = 0$ if $p > \dim X$ and $q > 0$

Def 2.23 Suppose $x \in X$. Define $\underline{H^i_x(X,F)} = \varinjlim_{\substack{x \in V \subseteq X \\ \text{open}}} H^i_{\bar{x} \cap V}(V, F)$. Then it is easy to see that

$H^i_x(X,F) \neq H^i_{\bar{x}}(X,F)$

$$\boxed{E_1^{p,q} = \bigoplus_{\operatorname{codim} \bar{x} = p} H^{p+q}_x(X,F).}$$

Prop 2.24 (Étale excision) Suppose $\varphi: Y \to X$ is étale, $Z \subseteq X$ s.t. $\underline{\varphi^{-1}(Z) = Z}$. For any $F \in Sh(\mathcal{C}_X)$, we have

$$\underline{H^i_Z(Y, \varphi^* F) = H^i_Z(X, F).}$$

Proof: The $Y \amalg X \setminus Z \to X$ is Nisnevich covering. Hence the sheaf condition of F gives a cartesian square

$$\begin{array}{ccc} H^0_Z(X,F) & = & H^0_Z(Y,F) \\ F(X) & \to & F(Y) \\ \downarrow & \square & \downarrow \\ F(X \setminus Z) & \to & F(Y \setminus Z), \end{array}$$

$H^0(Y, \varphi^* F) = F(Y)$

which shows the result for $i = 0$. The φ^* is exact and has a left adjoint $\varphi_!$, given by extension by zero

$$(\varphi_! F)(U) = \begin{cases} F(U) & \text{if } U \to Y \\ 0 & \text{else} \end{cases} \rightsquigarrow \text{take sheafification}$$

So φ^* preserves injective objects. So the statement for $i > 0$ follows (e.g. δ-functor) $\square$

Cor 2.25 $x \in X$ $H^i_x(X,F) = H^i_x(\operatorname{Spec} \mathcal{O}^h_{X,x}, F_x)$, $x \in \max \operatorname{Spec} \mathcal{O}^h_{X,x}$ $\qquad \operatorname{Spec} \mathcal{O}^h_{X,x} \to X$

Thm 2.26: $H^n(X,F) = 0 \quad \forall n > \dim(X)$

Proof: Let us prove by induction on $\dim X$. If $\dim X = 0$, X is a disjoint union of Spec of Henselian ring. So the statement follows.

Artinian $\Rightarrow$ complete

Now suppose the statement is true for any Y, $\dim Y < \dim X$. The exact sequence

$$\underline{H^i(\operatorname{Spec} \mathcal{O}^h_{X,x}, F_x)} \to H^i(\operatorname{Spec} \mathcal{O}^h_{X,x} \setminus x, F_x) \to H^{i+1}_x(\operatorname{Spec} \mathcal{O}^h_{X,x}, F_x) \to \underline{H^{i+1}(\operatorname{Spec} \mathcal{O}^h_{X,x}, F_x)}$$

shows that $\underline{H^i(\operatorname{Spec} \mathcal{O}^h_{X,x} \setminus x, F_x) = H^{i+1}_x(\operatorname{Spec} \mathcal{O}^h_{X,x}, F_x)}$, $i > 0$. By induction on dimension, we have $\underline{H^{n-1}(\operatorname{Spec} \mathcal{O}^h_{X,x} \setminus x, F_x) = 0}$ if $n > \dim \bar{x}$. So

$$E_1^{p,q} = \bigoplus_{\operatorname{codim} \bar{x} = p} H^{p+q}_x(\operatorname{Spec} \mathcal{O}^h_{X,x}, F_x) = 0 \qquad H^n_x(\operatorname{Spec} \mathcal{O}^h_{X,x}, F_x) = 0 \text{ if } n > \dim \bar{x}$$

if $p+q > \dim X$. So $H^n(X,F) = 0$, as desired. $\square$

Thm 2.27 Let $U, X \in Sm/S$ and $p: U \to X$ be a Nis covering. Denote the n-fold product $A \times_B A \times_B \cdots \times_B A$ by A^n_B. Then the complex of sheaves associated to the complex

$$\check{C}(U/X) = \cdots \to \mathbb{Z}(U^n_X) \xrightarrow{d_n} \cdots \to \mathbb{Z}(U \times_X U) \xrightarrow{d_2} \mathbb{Z}(U) \xrightarrow{d_1} \mathbb{Z}(X) \to 0$$

$b \mapsto a \mapsto 0$

is exact, where $d_n = \sum_i (-1)^{i-1} \mathbb{Z}(p_i)$, $p_i: U^n_X \to U^{n-1}_X$ is the map omitting i-th factor.

$(u_1, \dots, u_n) \mapsto (u_1, \dots, \hat{u}_i, \dots, u_n)$

Proof: We only need to check locally. So suppose $Y = \operatorname{Spec} A$ where A is Henselian local and $a \in Cor_S(Y, U^n_X) = \mathbb{Z}(U^n_X)(Y)$ s.t. $d_n(a) = 0$.

Define $T = \operatorname{Supp}(a)$ and $R = T \times_{X \times Y} (U \times Y)$, by Thm 2.20, T is a disjoint union

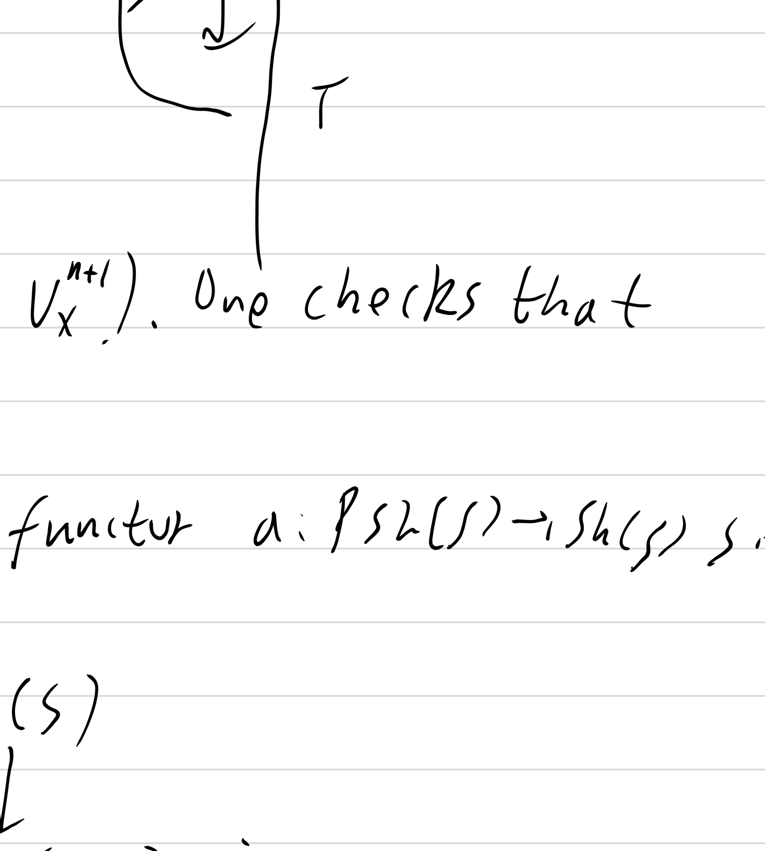
$$R \xrightarrow{Nis} T \subseteq Y \times_S U^n_X \to Y, \quad T \to Y \text{ finite surj}$$

of Spec of Henselians. The R is a Nisnevich covering of T so the map $R \to T$ admits a section $s: T \to R$, where T is both an open and closed subset of R.

We have a diagram of cartesian squares

$$\begin{array}{ccc} R^n_T & \longrightarrow & (U \times_S Y)^n_{X \times_S Y} \times_{X \times_S Y} (U \times_S Y \setminus (R \setminus T)) \\ \downarrow id^n \times s & & \downarrow j_{n+1} \\ R^{n+1}_T & \longrightarrow & (U \times_S Y)^{n+1}_{X \times_S Y} \\ \downarrow p_{n+1} & & \downarrow p_{n+1} \\ R^n_T & \longrightarrow & (U \times_S Y)^n_{X \times_S Y} \ni a \end{array}$$

$R \setminus (R \setminus T) = T$, $(\ )_*$, $(\ )^*$, $(\ )_!$



Then define $b = (j_{n+1*} \circ (p_{n+1} \circ j_{n+1})^*)(a) \in Cor_S(Y, U^{n+1}_X)$. One checks that $d_{n+1}(b) = a$. $\square$

Thm 2.28 There is a unique sheafification functor $a: PSh(S) \to Sh(S)$ s.t. the following diagram commutes:

$$\begin{array}{ccc} F \in PSh(S) & \xrightarrow{a} & Sh(S) \\ \downarrow & & \downarrow \\ PSh(Sm/S) & \xrightarrow{+} & Sh(Sm/S) \end{array}$$