Selected topics on category theory

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Condensation completion

A unitary fusion n-category $\mathcal{C}$ involves the following structures:

1. $\mathcal{C}$ is a $\mathbb{C}$-linear monoidal n-category with duals.
2. $\mathcal{C}$ is separable.
3. $\mathcal{C}$ carries a $*$-structure.

References:

[1] C. L. Douglas and D. J. Reutter, *Fusion 2-categories and a state-sum invariant for 4-manifolds*, `arXiv:1812.11933`.

[2] D. Gaiotto and T. Johnson-Freyd, *Condensations in higher categories*, `arXiv:1905.09566`.

[3] T. Johnson-Freyd, *On the classification of topological orders*, `arXiv:2003.06663`.

[4] L. Kong and H. Zheng, *Categories of quantum liquids I*, `arXiv:2011.02859`.

1. Condensation

Definition

Let $\mathcal{C}$ be an n-category and $X, Y \in \mathcal{C}$. For $n = 0$, a condensation $X \twoheadrightarrow Y$ is an equality $X = Y$. By induction on $n \geq 1$, a condensation $X \twoheadrightarrow Y$ is a pair of 1-morphisms $f : X \to Y$ and $g : Y \to X$ together with a condensation $f \circ g \twoheadrightarrow \mathrm{Id}_Y$. We say that Y is a condensate of X if there exists a condensation $X \twoheadrightarrow Y$.

Example

For $n = 0$, Y is a condensate of X if and only if $Y = X$.

Example

For $n = 1$, a condensation $X \twoheadrightarrow Y$ is a pair of 1-morphisms $Y \xrightarrow{g} X \xrightarrow{f} Y$ such that $f \circ g = \mathrm{Id}_Y$. Therefore, a condensate is nothing but a retract. Note that the 1-morphism $g \circ f : X \to X$ is idempotent.

Example (Trivial condensation)

$\mathrm{Id}_X : X \twoheadrightarrow X$.

Remark

In general, a condensation $X \twoheadrightarrow Y$ consists of the following data:

- a pair of 1-morphisms $Y \xrightarrow{g_1} X \xrightarrow{f_1} Y$,
- a pair of 2-morphisms $\mathrm{Id}_Y \xrightarrow{g_2} f_1 \circ g_1 \xrightarrow{f_2} \mathrm{Id}_Y$,
- a pair of 3-morphisms $\mathrm{Id}_{\mathrm{Id}_Y} \xrightarrow{g_3} f_2 \circ g_2 \xrightarrow{f_3} \mathrm{Id}_{\mathrm{Id}_Y}$, ...,
- a pair of n-morphisms $\mathrm{Id}_{\dots \mathrm{Id}_Y} \xrightarrow{g_n} f_{n-1} \circ g_{n-1} \xrightarrow{f_n} \mathrm{Id}_{\dots \mathrm{Id}_Y}$ such that $f_n \circ g_n = \mathrm{Id}_{\dots \mathrm{Id}_Y}$.

Remark

The relation of condensation is transitive. That is, giving a pair of condensations $X \twoheadrightarrow Y \twoheadrightarrow Z$ defined by $X \xrightarrow{f} Y \xrightarrow{f'} Z$, $X \xleftarrow{g} Y \xleftarrow{g'} Z$ and $f \circ g \twoheadrightarrow \mathrm{Id}_Y$, $f' \circ g' \twoheadrightarrow \mathrm{Id}_Z$, we have a condensation $X \twoheadrightarrow Z$ defined by $X \xrightarrow{f' \circ f} Z$, $X \xleftarrow{g \circ g'} Z$ and the composition $f' \circ f \circ g \circ g' \twoheadrightarrow f' \circ \mathrm{Id}_Y \circ g' \twoheadrightarrow \mathrm{Id}_Z$.

Definition

The walking n-condensation is the n-category $\spadesuit_n$ generated by

- a pair of objects X and Y,
- a pair of 1-morphisms $Y \xrightarrow{g_1} X \xrightarrow{f_1} Y$,
- a pair of 2-morphisms $\mathrm{Id}_Y \xrightarrow{g_2} f_1 \circ g_1 \xrightarrow{f_2} \mathrm{Id}_Y$,
- a pair of 3-morphisms $\mathrm{Id}_{\mathrm{Id}_Y} \xrightarrow{g_3} f_2 \circ g_2 \xrightarrow{f_3} \mathrm{Id}_{\mathrm{Id}_Y}$, ...,
- a pair of n-morphisms $\mathrm{Id}_{\dots\,\mathrm{Id}_Y} \xrightarrow{g_n} f_{n-1} \circ g_{n-1} \xrightarrow{f_n} \mathrm{Id}_{\dots\,\mathrm{Id}_Y}$ such that $f_n \circ g_n = \mathrm{Id}_{\dots\,\mathrm{Id}_Y}$.

The walking condensation monad $\clubsuit_n$ is the full subcategory on the object X.

By definition, a condensation in an n-category $\mathcal{C}$ is a functor $\spadesuit_n \to \mathcal{C}$.

Definition

A condensation monad in an n-category $\mathcal{C}$ is a functor $\clubsuit_n \to \mathcal{C}$.

Example (Trivial condensation monad)

A constant functor $\clubsuit_n \to \mathcal{C}$.

Example

The 1-category $\spadesuit_1$ consists of two objects X, Y and three nontrivial 1-morphisms $f : X \to Y$, $g : Y \to X$, $e : X \to X$ such that $e^2 = e = g \circ f$ and $f \circ g = \mathrm{Id}_Y$. Thus $\clubsuit_1$ consists of a single object X and a single nontrivial 1-morphisms $e : X \to X$ such that $e^2 = e$. Therefore, a condensation monad $\clubsuit_1 \to \mathcal{C}$ is simply an idempotent 1-morphism of $\mathcal{C}$.

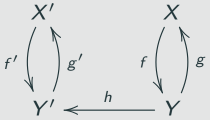
Definition

By induction on $n \geq 1$, we say that an n-category $\mathcal{C}$ is condensation complete or Karoubi complete if $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is condensation complete for all $X, Y \in \mathcal{C}$ and if every condensation monad in $\mathcal{C}$ extends to a condensation. A 0-category is always assumed to be condensation complete.

We use KarCat_n (resp. $E_m\mathrm{KarCat}_n$, KarCat_n^+, KarCat_n^R, etc.) to denote the full subcategory of Cat_n (resp. $E_m\mathrm{Cat}_n$, Cat_n^+, Cat_n^R) formed by condensation complete (E_m-monoidal, additive, R-linear, etc.) n-categories.

Definition

Let $\heartsuit_n$ denote the n-category generated by two copies of $\spadesuit_n$ and a 1-morphism $Y \to Y'$:

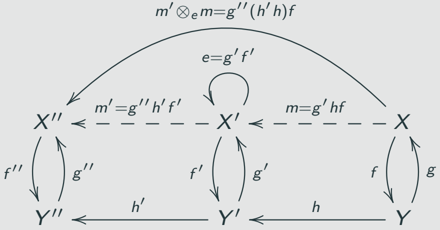


The walking n-condensation bimodule $\diamondsuit_n$ is the full subcategory of $\heartsuit_n$ on X and X'.

A condensation bimodule in an n-category $\mathcal{C}$ is a functor $\diamondsuit_n \to \mathcal{C}$.

Definition

The composition $m' \otimes_e m$ of two condensation bimodules m' and m over e is defined to be the condensate of the canonical condensation monad on $m'm$:



2. Idempotent completion

Definition

Let $\mathcal{C}$ be a 1-category. The idempotent completion or Karoubi completion of $\mathcal{C}$ is an idempotent complete 1-category $\mathrm{Kar}(\mathcal{C})$ equipped with a functor $\iota : \mathcal{C} \to \mathrm{Kar}(\mathcal{C})$ such that composition with ι induces an equivalence

$$\mathsf{Fun}(\mathrm{Kar}(\mathcal{C}), \mathcal{D}) \simeq \mathsf{Fun}(\mathcal{C}, \mathcal{D})$$

for any idempotent complete 1-category $\mathcal{D}$.

Remark

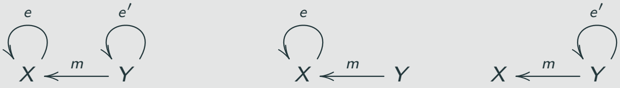
For $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}_1$, we have a functor $\mathsf{Fun}(\mathcal{C}, \mathcal{D}) \to \mathsf{Fun}(\mathcal{C}, \mathrm{Kar}(\mathcal{D})) \simeq \mathsf{Fun}(\mathrm{Kar}(\mathcal{C}), \mathrm{Kar}(\mathcal{D}))$. We obtain a functor

$$\mathrm{Kar} : \mathrm{Cat}_1 \to \mathrm{KarCat}_1, \quad \mathcal{C} \mapsto \mathrm{Kar}(\mathcal{C})$$

which is left adjoint to the inclusion $\mathrm{KarCat}_1 \hookrightarrow \mathrm{Cat}_1$.

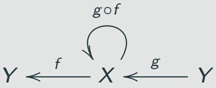
Definition

Let $e : X \to X$ and $e' : Y \to Y$ be idempotents. An e-e'-bimodule is 1-morphism $m : Y \to X$ such that $e \circ m = m = m \circ e'$. In the special case where $e' = \mathrm{Id}_Y$ we say that m is a left e-module. In the special case where $e = \mathrm{Id}_X$ we say that m is a right e'-module.



Example

If $f : X \to Y$ and $g : Y \to X$ exhibit Y as a retract of X, then g is a left $g \circ f$-module and f is a right $g \circ f$-module.



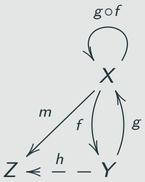
Proposition

Suppose that $f : X \to Y$ and $g : Y \to X$ exhibit Y as a retract of X. Then f (resp. g) exhibits Y as the colimit (resp. limit) of the circle diagram $X \circlearrowleft^{g \circ f}$.

Therefore, if an idempotent admits a retract, then the retract is unique up to unique isomorphism.

Proof.

Let $m : X \to Z$ be a 1-morphism such that $m \circ (g \circ f) = m$, i.e. m is a right $g \circ f$-module. We need to show that there exists a unique h rendering $m = h \circ f$. Indeed, h has to be $m \circ g$.

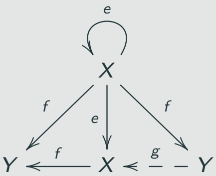


Proposition

Conversely, let $e : X \to X$ be an idempotent. If the circle diagram $\overset{e}{\circlearrowleft} X$ admits a colimit (resp. limit) Y, then Y is a retract of X.

Proof.

Suppose that $f : X \to Y$ exhibits Y as the colimit of the circle diagram. Then there exists a unique $g : Y \to X$ rendering $e = g \circ f$. By the universal property of a colimit, $\mathrm{Id}_Y = f \circ g$.



□

Theorem

Define a 1-category $\mathrm{Kar}(\mathcal{C})$ *of which an object is an idempotent* $e : X \to X$ *and a 1-morphism is a bimodule. Then the functor* $\iota : \mathcal{C} \to \mathrm{Kar}(\mathcal{C})$, $X \mapsto \mathrm{Id}_X$ *exhibits* $\mathrm{Kar}(\mathcal{C})$ *as the idempotent completion of* $\mathcal{C}$.

Proof.

First, $\mathrm{Kar}(\mathcal{C})$ is idempotent complete: an idempotent $d : e \to e$ admits a retract d.

Let $\mathcal{D}$ be an idempotent complete 1-category. Given a functor $F : \mathcal{C} \to \mathcal{D}$, we define a functor $\hat{F} : \mathrm{Kar}(\mathcal{C}) \to \mathcal{D}$ as follows. On object, $\hat{F}(e)$ is a retract of $F(e)$. On morphism, $\hat{F}(m)$ is induced by $F(m)$.

$$\begin{array}{ccc} F(e) \circlearrowright F(X) & \longrightarrow & \hat{F}(e) \\ \downarrow {\scriptstyle F(m)} & & \downarrow {\scriptstyle \hat{F}(m)} \\ F(e') \circlearrowright F(X') & \longrightarrow & \hat{F}(e') \end{array}$$

In this way, we obtain a functor $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathrm{Kar}(\mathcal{C}), \mathcal{D})$, $F \mapsto \hat{F}$. It is clear that $\hat{F} \circ \iota \simeq F$. By the uniqueness of retract, $\widehat{G \circ \iota} \simeq G$ for $G : \mathrm{Kar}(\mathcal{C}) \to \mathcal{D}$. □

Remark

If $\mathcal{C}$ is additive, then $\mathrm{Kar}(\mathcal{C})$ is also additive: the direct sum of two idempotents is also an idempotent and the sum of two bimodules is also a bimodule.

Remark

The functor $\iota : \mathcal{C} \to \mathrm{Kar}(\mathcal{C})$ is fully faithful.

Let $\mathcal{C}$ be a (small) 1-category and let $\mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}_0)$ be the category of presheaves. An object $F \in \mathcal{P}(\mathcal{C})$ is tiny or completely compact if the representable functor $\mathrm{Hom}_{\mathcal{P}(\mathcal{C})}(F, -) : \mathcal{P}(\mathcal{C}) \to \mathrm{Cat}_0$ preserves (small) colimits. We use $\mathrm{Cau}(\mathcal{C})$ to denote the full subcategory of $\mathcal{P}(\mathcal{C})$ formed by tiny objects and refer to it as the Cauchy completion of $\mathcal{C}$. The Yoneda embedding $j : \mathcal{C} \hookrightarrow \mathcal{P}(\mathcal{C})$ factors through $\mathrm{Cau}(\mathcal{C})$.

Theorem

The Yoneda embedding $j : \mathcal{C} \hookrightarrow \mathrm{Cau}(\mathcal{C})$ exhibits $\mathrm{Cau}(\mathcal{C})$ as the idempotent completion of $\mathcal{C}$.

Proof.

Since Cat_0 has (small) colimits, it is idempotent complete. Thus the embedding $\mathcal{C} \hookrightarrow \mathrm{Kar}(\mathcal{C})$ induces an equivalence $\mathcal{P}(\mathrm{Kar}(\mathcal{C})) \simeq \mathcal{P}(\mathcal{C})$ so that $\mathrm{Cau}(\mathrm{Kar}(\mathcal{C})) \simeq \mathrm{Cau}(\mathcal{C})$. Thus $j : \mathcal{C} \hookrightarrow \mathrm{Cau}(\mathcal{C})$ factors through $\mathrm{Kar}(\mathcal{C})$. It remains to show that every $F \in \mathrm{Cau}(\mathcal{C})$ is a retract of some $j(X)$.

Suppose $F = \varinjlim j(X_\alpha)$. Then $\mathrm{Hom}_{\mathcal{P}(\mathcal{C})}(F, F) \simeq \varinjlim \mathrm{Hom}_{\mathcal{P}(\mathcal{C})}(F, j(X_\alpha))$. Therefore, Id_F is the image of some $F \to j(X_\alpha)$, i,e. Id_F factors through $j(X_\alpha)$. That is, F is a retract of $j(X_\alpha)$. □

3. Condensation completion

Definition

Let $\mathcal{C}$ be a n-category. The condensation completion or Karoubi completion of $\mathcal{C}$ is a n-category $\mathrm{Kar}(\mathcal{C})$ equipped with a functor $\iota : \mathcal{C} \to \mathrm{Kar}(\mathcal{C})$ such that composition with ι induces an equivalence

$$\mathsf{Fun}(\mathrm{Kar}(\mathcal{C}), \mathcal{D}) \simeq \mathsf{Fun}(\mathcal{C}, \mathcal{D})$$

for any condensation complete n-category $\mathcal{D}$.

Remark

For $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}_n$, we have a functor $\mathsf{Fun}(\mathcal{C}, \mathcal{D}) \to \mathsf{Fun}(\mathcal{C}, \mathrm{Kar}(\mathcal{D})) \simeq \mathsf{Fun}(\mathrm{Kar}(\mathcal{C}), \mathrm{Kar}(\mathcal{D}))$. We obtain a functor

$$\mathrm{Kar} : \mathrm{Cat}_n \to \mathrm{KarCat}_n, \quad \mathcal{C} \mapsto \mathrm{Kar}(\mathcal{C})$$

which is left adjoint to the inclusion $\mathrm{KarCat}_n \hookrightarrow \mathrm{Cat}_n$.

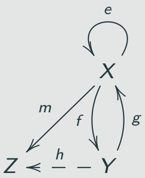
Proposition

Let $\mathcal{C}$ be an n-category whose hom categories are condensation complete. Suppose that a condensation monad $e : \clubsuit_n \to \mathcal{C}$ admits a condensate Y. Then Y is the colimit (resp. limit) of the diagram e.

Therefore, if a condensation monad admits a condensate, then the condensate is unique up to a contractible space of choices.

Proof.

Let $m : X \to Z$ be a right e-module. If $m = h \circ f$, then the condensation $f \circ g \twoheadrightarrow \mathrm{Id}_Y$ induces a condensation $m \circ g \twoheadrightarrow h$. Indeed, there is a condensation monad on $m \circ g$ taking the desired h as a condensate.



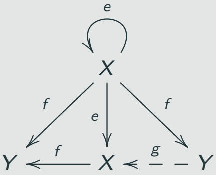
□

Proposition

Conversely, if a condensation monad $e : \clubsuit_n \to \mathcal{C}$ admits a colimit (resp. limit) Y, then Y is a condensate of e.

Proof.

Suppose that $f : X \to Y$ exhibits Y as the colimit of the diagram e. Then the trivial right e-module $e : X \to X$ determines a 1-morphism $g : Y \to X$. Moreover, the condensation $f \circ e \twoheadrightarrow f$ determines a condensation $f \circ g \twoheadrightarrow \mathrm{Id}_Y$.



□

Remark

If $\mathcal{C}$ is a condensation complete n-category, then $\mathsf{Fun}(\mathcal{D}, \mathcal{C})$ is also condensation complete for any n-category $\mathcal{D}$ becase $\mathsf{Fun}(\clubsuit_n, \mathsf{Fun}(\mathcal{D}, \mathcal{C})) \simeq \mathsf{Fun}(\mathcal{D}, \mathsf{Fun}(\clubsuit_n, \mathcal{C})) \simeq \mathsf{Fun}(\mathcal{D}, \mathsf{Fun}(\spadesuit_n, \mathcal{C})) \simeq \mathsf{Fun}(\spadesuit_n, \mathsf{Fun}(\mathcal{D}, \mathcal{C}))$.

Remark

We have a canonical equivalence for $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}_n$

$$\mathrm{Kar}(\mathcal{C} \times \mathcal{D}) \simeq \mathrm{Kar}(\mathcal{C}) \times \mathrm{Kar}(\mathcal{D})$$

because $\mathsf{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \simeq \mathsf{Fun}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}, \mathcal{E})) \simeq \mathsf{Fun}(\mathrm{Kar}(\mathcal{C}), \mathsf{Fun}(\mathrm{Kar}(\mathcal{D}), \mathcal{E})) \simeq \mathsf{Fun}(\mathrm{Kar}(\mathcal{C}) \times \mathrm{Kar}(\mathcal{D}), \mathcal{E})$ for $\mathcal{E} \in \mathrm{KarCat}_n$.

Remark

For $\mathcal{C} \in \mathrm{Cat}_n$, we have an n-category $\hat{\mathcal{C}}$ such that $O(\hat{\mathcal{C}}) = O(\mathcal{C})$ and $\mathsf{Hom}_{\hat{\mathcal{C}}}(X, Y) = \mathrm{Kar}(\mathsf{Hom}_{\mathcal{C}}(X, Y))$ for $X, Y \in \mathcal{C}$.

Note that $\mathrm{Kar}(\hat{\mathcal{C}}) \simeq \mathrm{Kar}(\mathcal{C})$. Therefore, to construct $\mathrm{Kar}(\mathcal{C})$ we may assume that $\mathcal{C}$ has condensation complete hom categories.

Theorem

Define an n-category $\mathrm{Kar}(\mathcal{C})$ *of which an object is a condensation monad* $e : \clubsuit_n \to \mathcal{C}$ *and a 1-morphism is a bimodule. Then the functor* $\iota : \mathcal{C} \to \mathrm{Kar}(\mathcal{C})$ *exhibits* $\mathrm{Kar}(\mathcal{C})$ *as the condensation completion of* $\mathcal{C}$.

Proof.

First, $\mathrm{Kar}(\mathcal{C})$ is condensation complete: a condensation monad $\clubsuit_n \to \mathrm{Kar}(\mathcal{C})$ admits a condensate given by the composition $\clubsuit_n \to \mathrm{Kar}(\mathcal{C}) \to \mathcal{C}$.

Let $\mathcal{D}$ be a condensation complete n-category. Given a functor $F : \mathcal{C} \to \mathcal{D}$, we define a functor $\hat{F} : \mathrm{Kar}(\mathcal{C}) \to \mathcal{D}$ as follows. On object, $\hat{F}(e)$ is a condensate of $F(e)$. On morphism, $\hat{F}(m)$ is induced by $F(m)$.

$$\begin{array}{ccc} F(e) \circlearrowright F(X) & \longrightarrow & \hat{F}(e) \\ \downarrow F(m) & & \downarrow \hat{F}(m) \\ F(e') \circlearrowright F(X') & \longrightarrow & \hat{F}(e') \end{array}$$

In this way, we obtain a functor $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathrm{Kar}(\mathcal{C}), \mathcal{D})$, $F \mapsto \hat{F}$. It is clear that $\hat{F} \circ \iota \simeq F$. By the uniqueness of condensate, $\widehat{G \circ \iota} \simeq G$ for $G : \mathrm{Kar}(\mathcal{C}) \to \mathcal{D}$. □

Remark

If $\mathcal{C}$ is additive, then $\mathrm{Kar}(\mathcal{C})$ is also additive. Therefore, the functor $\mathrm{Kar} : \mathrm{Cat}_n \to \mathrm{KarCat}_n$ restricts to a functor

$$\mathrm{Kar} : \mathrm{Cat}_n^+ \to \mathrm{KarCat}_n^+, \quad \mathcal{C} \mapsto \mathrm{Kar}(\mathcal{C})$$

which is left adjoint to the inclusion $\mathrm{KarCat}_n^+ \hookrightarrow \mathrm{Cat}_n^+$.

Remark

Recall that $\mathrm{Cat}_n^R = \mathsf{Fun}^+(B^{n+1}R, \mathrm{Cat}_n^+)$ and $\mathrm{KarCat}_n^R = \mathsf{Fun}^+(B^{n+1}R, \mathrm{KarCat}_n^+)$.
The functor $\mathrm{Kar} : \mathrm{Cat}_n^+ \to \mathrm{KarCat}_n^+$ induces a functor

$$\mathrm{Kar} : \mathrm{Cat}_n^R \to \mathrm{KarCat}_n^R, \quad \mathcal{C} \mapsto \mathrm{Kar}(\mathcal{C})$$

which is left adjoint to the inclusion $\mathrm{KarCat}_n^R \hookrightarrow \mathrm{Cat}_n^R$.

Let $\mathcal{C}$ be a (small) n-category whose hom categories are condensation complete. Let $\mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{KarCat}_{n-1})$. An object $F \in \mathcal{P}(\mathcal{C})$ is tiny or completely compact if the representable functor $\mathrm{Hom}_{\mathcal{P}(\mathcal{C})}(F, -) : \mathcal{P}(\mathcal{C}) \to \mathrm{KarCat}_{n-1}$ preserves (small) colimits. We use $\mathrm{Cau}(\mathcal{C})$ to denote the full subcategory of $\mathcal{P}(\mathcal{C})$ formed by tiny objects and refer to it as the Cauchy completion of $\mathcal{C}$. The Yoneda embedding $j : \mathcal{C} \hookrightarrow \mathcal{P}(\mathcal{C})$ factors through $\mathrm{Cau}(\mathcal{C})$.

Theorem

The Yoneda embedding $j : \mathcal{C} \hookrightarrow \mathrm{Cau}(\mathcal{C})$ exhibits $\mathrm{Cau}(\mathcal{C})$ as the condensation completion of $\mathcal{C}$.

Proof.

Since KarCat_{n-1} has (small) colimits, it is condensation complete. Thus the embedding $\mathcal{C} \hookrightarrow \mathrm{Kar}(\mathcal{C})$ induces an equivalence $\mathcal{P}(\mathrm{Kar}(\mathcal{C})) \simeq \mathcal{P}(\mathcal{C})$ so that $\mathrm{Cau}(\mathrm{Kar}(\mathcal{C})) \simeq \mathrm{Cau}(\mathcal{C})$. Thus $j : \mathcal{C} \hookrightarrow \mathrm{Cau}(\mathcal{C})$ factors through $\mathrm{Kar}(\mathcal{C})$. It remains to show that every $F \in \mathrm{Cau}(\mathcal{C})$ is a condensate of some $j(X)$.

Suppose $F = \varinjlim j(X_\alpha)$. Then $\mathrm{Hom}_{\mathcal{P}(\mathcal{C})}(F, F) \simeq \varinjlim \mathrm{Hom}_{\mathcal{P}(\mathcal{C})}(F, j(X_\alpha))$. Let $\mathcal{D} \subset \mathrm{Hom}_{\mathcal{P}(\mathcal{C})}(F, F)$ be the full subcategory formed by the images of $\mathrm{Hom}_{\mathcal{P}(\mathcal{C})}(F, j(X_\alpha))$. Then $\mathrm{Kar}(\mathcal{D}) = \mathrm{Hom}_{\mathcal{P}(\mathcal{C})}(F, F)$. Therefore, there exists $g : F \to j(X_\alpha)$ such that Id_F is a condensate of the image of g. That is, F is a condensate of $j(X_\alpha)$. □

Let $\mathcal{C}$ be a (small) condensation complete monoidal n-category. We use $\Sigma\mathcal{C}$ to denote $\mathrm{Kar}(B\mathcal{C})$. Then the functor

$$\Sigma : E_m\mathrm{KarCat}_n \to E_{m-1}\mathrm{KarCat}_{n+1}$$

is left adjoint to

$$\Omega : E_{m-1}\mathrm{KarCat}_{n+1} \to E_m\mathrm{KarCat}_n.$$

Note that $\mathcal{P}(B\mathcal{C}) = \mathrm{Fun}(B\mathcal{C}^{\mathrm{rev}}, \mathrm{KarCat}_n) = \mathrm{RMod}_{\mathcal{C}}(\mathrm{KarCat}_n)$. The embedding $\Sigma\mathcal{C} \hookrightarrow \mathrm{RMod}_{\mathcal{C}}(\mathrm{KarCat}_n)$ is given by $X \mapsto \mathrm{Hom}_{\Sigma\mathcal{C}}(\bullet, X)$.

Theorem

Let $\mathcal{C}$ be a (small) condensation complete monoidal n-category. The functor $\mathrm{Hom}_{\Sigma\mathcal{C}}(\bullet, -) : \Sigma\mathcal{C} \to \mathrm{RMod}_{\mathcal{C}}(\mathrm{KarCat}_n)$ is fully faithful. Moreover, the following conditions are equivalent for an object $\mathcal{M} \in \mathrm{RMod}_{\mathcal{C}}(\mathrm{KarCat}_n)$:

1. *$\mathcal{M}$ belongs to the essential image.*
2. *The functor $\mathrm{Fun}_{\mathcal{C}^{\mathrm{rev}}}(\mathcal{M}, -) : \mathrm{RMod}_{\mathcal{C}}(\mathrm{KarCat}_n) \to \mathrm{KarCat}_n$ preserves (small) colimits.*
3. *The evaluation functor $\mathrm{Fun}_{\mathcal{C}^{\mathrm{rev}}}(\mathcal{M}, \mathcal{C}) \times \mathcal{M} \to \mathcal{C}$ exhibits the left $\mathcal{C}$-module $\mathrm{Fun}_{\mathcal{C}^{\mathrm{rev}}}(\mathcal{M}, \mathcal{C})$ dual to $\mathcal{M}$.*
4. *$\mathcal{M}$ has a left dual in $\mathrm{LMod}_{\mathcal{C}}(\mathrm{KarCat}_n)$.*

Proof.

(1) $\Leftrightarrow$ (2) is due to the previous theorem. (2) $\Rightarrow$ (3) The unit map is $\bullet \to \mathrm{Fun}_{\mathcal{C}^{\mathrm{rev}}}(\mathcal{M}, \mathcal{M}) \simeq \mathcal{M} \times_{\mathcal{C}} \mathrm{Fun}_{\mathcal{C}^{\mathrm{rev}}}(\mathcal{M}, \mathcal{C})$. (3) $\Rightarrow$ (4) is trivial. (4) $\Rightarrow$ (2) $\mathrm{Fun}_{\mathcal{C}^{\mathrm{rev}}}(\mathcal{M}, -) \simeq - \times_{\mathcal{C}} \mathcal{M}^L$. □

4. Duality

Definition

A condensation $X \twoheadrightarrow Y$ is unital if the 2-morphism $f \circ g \to \mathrm{Id}_Y$ exhibits $f : X \to Y$ left dual to $g : Y \to X$. A condensation $X \twoheadrightarrow Y$ is counital if the 2-morphism $\mathrm{Id}_Y \to f \circ g$ exhibits $f : X \to Y$ right dual to $g : Y \to X$.

Remark

If a condensation $X \twoheadrightarrow Y$ is unital, then the unit map $u : \mathrm{Id}_X \to g \circ f$ supplies a unit for the nonunital algebra $g \circ f$ in $\mathrm{Hom}_{\mathcal{C}}(X, X)$. If a condensation $X \twoheadrightarrow Y$ is counital, then the counit map $g \circ f \to \mathrm{Id}_X$ supplies a counit for the nonunital coalgebra $g \circ f$.

Proposition

Let $f : X \twoheadrightarrow Y$ be a condensation. If the 1-morphism $f : X \to Y$ admits a right adjoint f^R, then the counit map $v : f \circ f^R \to \mathrm{Id}_Y$ extends to a condensation (in particular, the condensation $f : X \twoheadrightarrow Y$ can be modified to a unital one).

Proof.

Let u' be the composition $\mathrm{Id}_Y \xrightarrow{g_1} f \circ g \xrightarrow{u} f \circ f^R \circ f \circ g \xrightarrow{f_1} f \circ f^R$. Then $v \circ u' \simeq f_1 \circ g_1$ condense to $\mathrm{Id}_{\mathrm{Id}_Y}$. □

Remark

In an n-category $\mathcal{C}$ that has duals, we may assume a condensation $f : X \twoheadrightarrow Y$ is given by consecutive counit maps $v_1 : f \circ f^R \to \mathrm{Id}_Y$, $v_2 : v_1 \circ v_1^R \to \mathrm{Id}_{\mathrm{Id}_Y}$, ..., such that v_{n-1} is a retraction.

Theorem

Let $\mathcal{C}$ be an n-category whose hom categories are condensation complete. If $\mathcal{C}$ has duals, then so is $\mathrm{Kar}(\mathcal{C})$.

Proof.

We need to show that every 1-morphism $\mu : Y \to Y'$ in $\mathrm{Kar}(\mathcal{C})$ admits a right dual. Choose condensations $f : X \twoheadrightarrow Y$ and $f' : X' \twoheadrightarrow Y'$ where $X, X' \in \mathcal{C}$.

First, the 1-morphism $(g \circ f)^R \circ g$ supports a condensation monad, condensing to f^R. Similarly, f admits a left dual.

In general, $f' \circ (f'^R \circ \mu \circ f) \circ f^L$ condense to μ. Thus $f \circ (f'^R \circ \mu \circ f)^R \circ f'^R$ supports a condensation monad, condensing to the desired μ^R. □