Selected topics on category theory

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Higher categories

A unitary fusion n-category $\mathcal{C}$ involves the following structures:

1. $\mathcal{C}$ is an n-category.
2. $\mathcal{C}$ is a monoidal n-category.
3. $\mathcal{C}$ has duals.
4. $\mathcal{C}$ is additive.
5. $\mathcal{C}$ is $\mathbb{C}$-linear.
6. $\mathcal{C}$ is semisimple.
7. $\mathcal{C}$ carries a $*$-structure.

1. n-Categories

Definition

A 0-category is a set $\mathcal{C}$ whose members are called objects.

A functor $F : \mathcal{C} \to \mathcal{D}$ between two 0-categories is a map of sets.

The Cartesian product $\mathcal{C} \times \mathcal{D}$ of two 0-categories is also a 0-category.

A singleton is called a trivial 0-category.

For 0-categories $\mathcal{C}$ and $\mathcal{D}$, the collection of functors $F : \mathcal{C} \to \mathcal{D}$ form a 0-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Definition

A 1-category $\mathcal{C}$ consists of the following data:

- A set $O(\mathcal{C})$ whose members are called objects. We use $X \in \mathcal{C}$ to denote the fact that X is an object of $\mathcal{C}$.
- For any $X, Y \in \mathcal{C}$, a 0-category $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ whose members are called 1-morphisms from X to Y. We use the notation $f : X \to Y$ or $X \xrightarrow{f} Y$ to denote a 1-morphism.
- For any $X, Y, Z \in \mathcal{C}$, a functor $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$, $(g, f) \mapsto g \circ f$, called the composition law.

These data satisfy the following conditions:

- (Unity) For any $X \in \mathcal{C}$ there exists a 1-morphism $\mathrm{Id}_X : X \to X$, called the identity 1-morphism of X, such that $f \circ \mathrm{Id}_X = f = \mathrm{Id}_Y \circ f$ for any $f : X \to Y$.
- (Associativity) For any $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$, $(h \circ g) \circ f = h \circ (g \circ f)$.

Example

A 0-category can be viewed as a 1-category whose 1-morphisms are only the identity 1-morphisms. In particular, a trivial 0-category (i.e. a singleton) is called a trivial 1-category.

Example

The collection of (small) 0-categories and functors form a 1-category Cat_0.

The Cartesian product $\mathcal{C} \times \mathcal{D}$ of two 1-categories is defined by

$$O(\mathcal{C} \times \mathcal{D}) = O(\mathcal{C}) \times O(\mathcal{D}),$$

$$\mathrm{Hom}_{\mathcal{C} \times \mathcal{D}}((X, Y), (X', Y')) = \mathrm{Hom}_{\mathcal{C}}(X, X') \times \mathrm{Hom}_{\mathcal{D}}(Y, Y').$$

More precisely, an object of $\mathcal{C} \times \mathcal{D}$ is a pair of objects (X, Y) and a 1-morphism is a pair of 1-morphisms (f, g).

Given a 1-category $\mathcal{C}$, we use $\mathcal{C}^{\mathrm{op}}$ to denote the 1-category obtained by reversing the 1-morphisms of $\mathcal{C}$, i.e. $O(\mathcal{C}^{\mathrm{op}}) = O(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$.

Definition

A functor $F : \mathcal{C} \to \mathcal{D}$ between two 1-categories consists of the following data:

- A map of sets $F : O(\mathcal{C}) \to O(\mathcal{D})$.
- For any $X, Y \in \mathcal{C}$, a functor $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$, $f \mapsto F(f)$.

These data satisfy the following conditions:

- For any $X \in \mathcal{C}$, $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$.
- For any $X \xrightarrow{f} Y \xrightarrow{g} Z$, $F(g \circ f) = F(g) \circ F(f)$.

Definition

Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors between 1-categories. A natural transformation $\xi : F \to G$ consists of the following data:

- For any $X \in \mathcal{C}$, a 1-morphisms $\xi_X : F(X) \to G(X)$.

The following diagram is commutative for any 1-morphism $f : X \to Y$ of $\mathcal{C}$:

$$\begin{tikzcd} F(X) \arrow[r, "\xi_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\xi_Y"] & G(Y). \end{tikzcd}$$

For 1-categories $\mathcal{C}$ and $\mathcal{D}$, the collection of functors $F : \mathcal{C} \to \mathcal{D}$ and natural transformations form a 1-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Informally speaking, an *n-category* $\mathcal{C}$ consists of the following data:

- A set $O(\mathcal{C})$ whose members are called objects.
- For any $X, Y \in \mathcal{C}$, an $(n-1)$-category $\mathrm{Hom}_{\mathcal{C}}(X, Y)$.
- For any $X, Y, Z \in \mathcal{C}$, a functor $\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$, called the composition law.
- A collection of coherence relations and conditions.

An object of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is called a 1-morphism from X to Y. A k-morphism of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is called a $(k+1)$-morphism of $\mathcal{C}$.

We say that two n-morphisms f and g of $\mathcal{C}$ are equivalent and denote $f \simeq g$ if $f = g$. By induction on $1 \leq k \leq n$, we say that a k-morphism $f : X \to Y$ is invertible if there exists a k-morphism $g : Y \to X$ such that $g \circ f \simeq \mathrm{Id}_X$ and $f \circ g \simeq \mathrm{Id}_Y$. We say that two objects or $(k-1)$-morphisms X and Y are equivalent and denote $X \simeq Y$ if there is an invertible 1-morphism or k-morphism between them.

The Cartesian product $\mathcal{C} \times \mathcal{D}$ of two n-categories is defined by

$$O(\mathcal{C} \times \mathcal{D}) = O(\mathcal{C}) \times O(\mathcal{D}),$$

$$\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((X,Y),(X',Y')) = \mathrm{Hom}_{\mathcal{C}}(X,X') \times \mathrm{Hom}_{\mathcal{D}}(Y,Y').$$

More precisely, an object of $\mathcal{C} \times \mathcal{D}$ is a pair of objects (X,Y) and a k-morphism is a pair of k-morphisms (f,g).

An $(n-1)$-category can be viewed as an n-category whose n-morphisms are only the identity n-morphisms. In particular, a trivial 0-category (i.e. a singleton) is called a trivial n-category. A trivial n-category consists of a single object and a single k-morphism for $1 \leq k \leq n$.

For an n-category $\mathcal{C}$, we use $\mathcal{C}^{\mathrm{op}k}$ to denote the n-category obtained by reversing all the k-morphisms. In particular, $\mathcal{C}^{\mathrm{op}}$ denotes $\mathcal{C}^{\mathrm{op}1}$.

Definition

A 2-category $\mathcal{C}$ consists of the following data:

- A set $O(\mathcal{C})$ whose members are called objects.
- For any $X, Y \in \mathcal{C}$, a 1-category $\mathrm{Hom}_{\mathcal{C}}(X, Y)$.
- For any $X, Y, Z \in \mathcal{C}$, a functor $\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$.
- For any $X \in \mathcal{C}$, a 1-morphism $\mathrm{Id}_X : X \to X$.
- Natural 2-isomorphisms $\mathrm{Id}_X \circ f \simeq f \simeq f \circ \mathrm{Id}_Y$ for 1-morphisms $f : X \to Y$ (we will assume without loss of generality that they are the identities).
- A natural 2-isomorphism $(h \circ g) \circ f \simeq h \circ (g \circ f)$ for $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$.

These data satisfy the triangle and pentagon conditions.

For $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W \xrightarrow{i} V$,

$$\begin{tikzcd}
& g \circ f & \\
(g \circ \mathrm{Id}_Y) \circ f \arrow[ur] \arrow[rr] & & g \circ (\mathrm{Id}_Y \circ f), \arrow[ul]
\end{tikzcd}$$

$$\begin{tikzcd}
& (i \circ h) \circ (g \circ f) \arrow[dr] & \\
((i \circ h) \circ g) \circ f \arrow[ur] \arrow[d] & & i \circ (h \circ (g \circ f)) \\
(i \circ (h \circ g)) \circ f \arrow[r] & i \circ ((h \circ g)) \circ f). \arrow[ur] &
\end{tikzcd}$$

Definition

A functor $F : \mathcal{C} \to \mathcal{D}$ between 2-categories consists of the following data:

- A map of sets $F : O(\mathcal{C}) \to O(\mathcal{D})$.
- For any $X, Y \in \mathcal{C}$, a functor $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$.
- For any $X \in \mathcal{C}$, a 2-isomorphism $F(\mathrm{Id}_X) \simeq \mathrm{Id}_{F(X)}$ (we will assume without loss of generality that it is the identity).
- A natural 2-isomorphism $F(g \circ f) \simeq F(g) \circ F(f)$ for $X \xrightarrow{f} Y \xrightarrow{g} Z$.

These data satisfy the following conditions for $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$:

$$\begin{tikzcd}
F(f \circ \mathrm{Id}_Y) \arrow[r] \arrow[d] & F(f) \circ F(\mathrm{Id}_Y) \arrow[d] \\
F(f) & F(f) \circ \mathrm{Id}_{F(Y)}, \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
F(h \circ g \circ f) \arrow[r] \arrow[d] & F(h \circ g) \circ F(f) \arrow[d] \\
F(h) \circ F(g \circ f) \arrow[r] & F(h) \circ F(g) \circ F(f).
\end{tikzcd}$$

Definition

Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors between 2-categories. A natural transformation $\xi : F \to G$ consists of the following data:

- For any $X \in \mathcal{C}$, a 1-morphism $\xi_X : F(X) \to G(X)$.
- A natural 2-isomorphism $\theta_f : \xi_Y \circ F(f) \simeq G(f) \circ \xi_X$ for $f : X \to Y$.

These data satisfy the following conditions for $X \xrightarrow{f} Y \xrightarrow{g} Z$:

$$
\begin{tikzcd}
\xi_X \circ F(\mathrm{Id}_X) \arrow[r, "\theta"] \arrow[d] & G(\mathrm{Id}_X) \circ \xi_X \arrow[d] \\
\xi_X \circ \mathrm{Id}_{F(X)} \arrow[dr] & \mathrm{Id}_{G(X)} \circ \xi_X \arrow[d] \\
 & \xi_X,
\end{tikzcd}
\qquad
\begin{tikzcd}
\xi_Z \circ F(g \circ f) \arrow[r, "\theta"] \arrow[d] & G(g \circ f) \circ \xi_X \arrow[d] \\
\xi_Z \circ F(g) \circ F(f) \arrow[d, "\theta"] & G(g) \circ G(f) \circ \xi_X \\
G(g) \circ \xi_Y \circ F(f). \arrow[ur, "\theta"] &
\end{tikzcd}
$$

Definition

Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors between 2-categories and $\xi, \zeta : F \to G$ be natural transformations. A modification or 2-natural transformation $\eta : \xi \to \zeta$ consists of the following data:

- For any $X \in \mathcal{C}$, a 2-morphism $\eta_X : \xi_X \to \zeta_X$.

These data satisfy the following conditions for $X \xrightarrow{f} Y$:

$$\begin{tikzcd} \xi_Y \circ F(f) \arrow[r] \arrow[d, "\eta_Y"'] & G(f) \circ \xi_X \arrow[d, "\eta_X"] \\ \zeta_Y \circ F(f) \arrow[r] & G(f) \circ \zeta_X . \end{tikzcd}$$

In general, a functor $F : \mathcal{C} \to \mathcal{D}$ between two n-categories consists of the following data:

- A map of sets $F : O(\mathcal{C}) \to O(\mathcal{D})$.
- For any $X, Y \in \mathcal{C}$, a functor $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$.
- A collection of coherence relations and conditions.

A natural transformation $\xi : F \to G$ between functors consists of the following data:

- For any $X \in \mathcal{C}$, a 1-morphism $\xi_X : F(X) \to G(X)$.
- A collection of coherence relations and conditions.

A k-natural transformation $\eta : \xi \to \zeta$ between $(k-1)$-natural transformations where $2 \leq k \leq n$ consists of the following data:

- For any $X \in \mathcal{C}$, a k-morphism $\eta_X : \xi_X \to \zeta_X$.
- A collection of coherence relations and conditions.

For n-categories $\mathcal{C}$ and $\mathcal{D}$, the collection of functors $F : \mathcal{C} \to \mathcal{D}$ and (higher) natural transformations form an n-category $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$.

The collection of (small) n-categories and functors form an $(n+1)$-category Cat_n such that $\mathsf{Hom}_{\mathrm{Cat}_n}(\mathcal{C}, \mathcal{D}) = \mathsf{Fun}(\mathcal{C}, \mathcal{D})$.

We say that a functor or (higher) natural transformation is invertible if it is an invertible morphism of Cat_n. We say that two n-categories are equivalent if they are equivalent objects of Cat_n. An invertible functor is called an equivalence. An invertible (higher) natural transformation is also called a (higher) natural equivalence.

We say that an n-category is contractible if it is equivalent to a trivial n-category.

Example

For two rings A and B, the A-B-bimodules and bimodules maps form a 1-category $\mathrm{BMod}_{A|B}$.

The collection of rings form a 2-category Mor where $\mathrm{Hom}_{\mathrm{Mor}}(A, B) = \mathrm{BMod}_{A|B}$. The composition $\circ : \mathrm{Hom}_{\mathrm{Mor}}(B, C) \times \mathrm{Hom}_{\mathrm{Mor}}(A, B) \to \mathrm{Hom}_{\mathrm{Mor}}(A, C)$ is given by $(M, N) \mapsto N \otimes_B M$.

2. Monoidal n-categories

Let $\mathcal{B}$ be a 1-category with a single object $\bullet$ and let $\mathcal{C} = \mathrm{Hom}_{\mathcal{B}}(\bullet, \bullet)$. Then $\mathcal{C}$ is a monoid with multiplication $\circ : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and unit $\mathrm{Id}_\bullet$. Conversely, every monoid is obtained in this way.

Let $\mathcal{B}$ be a 2-category with a single object $\bullet$ and let $\mathcal{C} = \mathrm{Hom}_{\mathcal{B}}(\bullet, \bullet)$. Then $\mathcal{C}$ is a monoidal 1-category with tensor product $\circ : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and tensor unit $\mathrm{Id}_\bullet$. Conversely, every monoidal 1-category is obtained in this way.

Definition

A monoidal n-category or E_1-monoidal n-category is a pair $(\mathcal{C}, B\mathcal{C})$ where $B\mathcal{C}$ is an $(n+1)$-category with a single object $\bullet$ and $\mathcal{C} = \mathrm{Hom}_{B\mathcal{C}}(\bullet, \bullet)$. By abusing terminology, we also refer to $\mathcal{C}$ as a monoidal n-category. The identity 1-morphism $\mathrm{Id}_\bullet$ is referred to as the tensor unit of $\mathcal{C}$ and denoted by $1_{\mathcal{C}}$. The functor $\circ : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is usually denoted by $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

A monoidal 0-category is a monoid. A monoidal 1-category is the one in the usual sense.

Definition

An E_0-monoidal n-category is a pair $(\mathcal{C}, 1_\mathcal{C})$ where $\mathcal{C}$ is an n-category and $1_\mathcal{C} \in \mathcal{C}$ is a distinguished object. When $1_\mathcal{C}$ is clear from the context, we use $\mathcal{C}$ to denote an E_0-monoidal n-category.

An E_0-monoidal functor $(\mathcal{C}, 1_\mathcal{C}) \to (\mathcal{D}, 1_\mathcal{D})$ between E_0-monoidal n-categories is a functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(1_\mathcal{C}) = 1_\mathcal{D}$. An E_0-monoidal (higher) natural transformation is a (higher) natural transformation η such that $\eta_{1_\mathcal{C}} = \mathrm{Id}$.

We use $E_0\mathrm{Cat}_n$ to denote the $(n+1)$-category formed by the E_0-monoidal n-categories, functors and (higher) natural transformations and use $\mathrm{Fun}^{E_0}((\mathcal{C}, 1_\mathcal{C}), (\mathcal{D}, 1_\mathcal{D}))$ to denote $\mathrm{Hom}_{E_0\mathrm{Cat}_n}((\mathcal{C}, 1_\mathcal{C}), (\mathcal{D}, 1_\mathcal{D}))$ which is a subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

For a monoidal n-category $\mathcal{C}$, the pair $(B\mathcal{C}, \bullet)$ defines an object of $E_0\mathrm{Cat}_{n+1}$. We use $E_1\mathrm{Cat}_n$ to denote the full subcategory of $E_0\mathrm{Cat}_{n+1}$ consisting of $(B\mathcal{C}, \bullet)$ for all monoidal n-categories $\mathcal{C}$. It turns out that $E_1\mathrm{Cat}_n$ is an $(n+1)$-category rather than an $(n+2)$-category. We use $\mathrm{Fun}^{E_1}(\mathcal{C}, \mathcal{D})$ to denote $\mathrm{Fun}^{E_0}(B\mathcal{C}, B\mathcal{D})$, the n-category formed by monoidal functors and invertible monoidal (higher) natural transformations.

We have a pair of functors

$$B : E_1\mathrm{Cat}_n \to E_0\mathrm{Cat}_{n+1}, \quad \mathcal{C} \mapsto (B\mathcal{C}, \bullet),$$

$$\Omega : E_0\mathrm{Cat}_{n+1} \to E_1\mathrm{Cat}_n, \quad (\mathcal{D}, 1_\mathcal{D}) \mapsto \mathrm{Hom}_\mathcal{D}(1_\mathcal{D}, 1_\mathcal{D})$$

where B is called the delooping functor and Ω is called the looping functor. By definition, $\mathrm{Fun}^{E_1}(\mathcal{C}, \Omega(\mathcal{D}, 1_\mathcal{D})) = \mathrm{Fun}^{E_0}((B\mathcal{C}, \bullet), B\Omega(\mathcal{D}, 1_\mathcal{D}))$. Therefore, B is left adjoint to Ω:

$$\mathrm{Fun}^{E_0}((B\mathcal{C}, \bullet), (\mathcal{D}, 1_\mathcal{D})) = \mathrm{Fun}^{E_1}(\mathcal{C}, \Omega(\mathcal{D}, 1_\mathcal{D})).$$

Example

Let $\mathcal{C}, \mathcal{D} \in E_1\mathrm{Cat}_0$ be two monoids. A monoidal functor $\mathcal{C} \to \mathcal{D}$ is by definition an E_0-monoidal functor $F : (B\mathcal{C}, \bullet) \to (B\mathcal{D}, \bullet)$ which is nothing but a homomorphism of monoids.

Let $\xi : F \to G$ be an E_0-monoidal natural transformation. By definition, $F(\bullet) = G(\bullet) = \bullet$ and $\xi_\bullet = \mathrm{Id}_\bullet$ so that $F(f) = G(f)$ for $f \in \mathcal{C}$. That is, $\xi = \mathrm{Id}_F$. Therefore, $\mathrm{Fun}^{E_1}(\mathcal{C}, \mathcal{D})$ is a 0-category rather than a 1-category. Therefore, $E_1\mathrm{Cat}_0$ is the 1-category of monoids and homomorphisms.

Example

Let $\mathcal{C}, \mathcal{D} \in E_1\mathrm{Cat}_1$ be two monoidal 1-categories. A monoidal functor $F : \mathcal{C} \to \mathcal{D}$ is by definition an E_0-monoidal functor $F : (B\mathcal{C}, \bullet) \to (B\mathcal{D}, \bullet)$ which is a monoidal functor in the usual sense.

Let $\xi : F \to G$ be an E_0-monoidal natural transformation. By definition, $F(\bullet) = G(\bullet) = \bullet$ and $\xi_\bullet = \mathrm{Id}_\bullet$ so that ξ is nothing but a monoidal natural isomorphism $F(f) \simeq G(f)$ in the usual sense.

By definition, an E_0-monoidal 2-natural transformation $\xi \to \zeta$ is an identity. Therefore, $E_1\mathrm{Cat}_1$ is the 2-category of monoidal 1-categories, monoidal functors and monoidal natural isomorphisms.

Definition

By induction on $m \geq 2$, an E_m-monoidal n-category is pair $(\mathcal{C}, B\mathcal{C})$ where $B\mathcal{C}$ is an E_{m-1}-monoidal $(n+1)$-category with a single object $\bullet$ and $\mathcal{C} = \mathrm{Hom}_{B\mathcal{C}}(\bullet, \bullet)$. By abusing terminology, we also refer to $\mathcal{C}$ as an E_m-monoidal n-category.

Note that an E_m-monoidal n-category consists of a finite series $(\mathcal{C}, B\mathcal{C}, B^2\mathcal{C}, \dots, B^m\mathcal{C})$.

For an E_m-monoidal n-category $\mathcal{C}$ where $m \geq 1$, the pair $(B^m\mathcal{C}, \bullet)$ defines an object of $E_0\mathrm{Cat}_{n+m}$. We use $E_m\mathrm{Cat}_n$ to denote the full subcategory of $E_0\mathrm{Cat}_{n+m}$ consisting of $(B^m\mathcal{C}, \bullet)$ for all E_m-monoidal n-categories $\mathcal{C}$. It turns out that $E_m\mathrm{Cat}_n$ is an $(n+1)$-category rather than an $(n+m+1)$-category. We use $\mathrm{Fun}^{E_m}(\mathcal{C}, \mathcal{D})$ to denote $\mathrm{Fun}^{E_0}(B^m\mathcal{C}, B^m\mathcal{D})$, the n-category formed by E_m-monoidal functors and invertible E_m-monoidal (higher) natural transformations.

We have a pair of functors $B : E_m\mathrm{Cat}_n \to E_{m-1}\mathrm{Cat}_{n+1}$, $\mathcal{C} \mapsto B\mathcal{C}$, called the delooping functor, and $\Omega : E_{m-1}\mathrm{Cat}_{n+1} \to E_m\mathrm{Cat}_n$, $\mathcal{D} \mapsto \mathrm{Hom}_{\mathcal{D}}(1_{\mathcal{D}}, 1_{\mathcal{D}})$, called the looping functor. By definition, B is left adjoint to Ω:

$$\mathrm{Fun}^{E_{m-1}}(B\mathcal{C}, \mathcal{D}) = \mathrm{Fun}^{E_m}(\mathcal{C}, \Omega\mathcal{D}).$$

Example

Let $(\mathcal{C}, B\mathcal{C})$ be an E_2-monoidal 0-category. That is, $B\mathcal{C}$ is a monoidal 1-category with a single object $\bullet$ and $\mathcal{C} = \mathrm{Hom}_{B\mathcal{C}}(\bullet, \bullet)$.

We have a tensor product functor $\otimes : B\mathcal{C} \times B\mathcal{C} \to B\mathcal{C}$ so that we have $(X \circ X') \otimes (Y \circ Y') = (X \otimes Y) \circ (X' \otimes Y')$ for $X, X, Y, Y' \in \mathcal{C}$.
The identity natural isomorphism $\bullet \otimes \bullet \simeq \bullet \simeq \bullet \otimes \bullet$ means that $\mathrm{Id}_\bullet \otimes X = X = X \otimes \mathrm{Id}_\bullet$.

Taking $Y = X' = \mathrm{Id}_\bullet$ yields $X \otimes Y' = X \circ Y'$. Taking $X = Y' = \mathrm{Id}_\bullet$ yields $X' \otimes Y = Y \circ X'$. Therefore, we obtain

$$X \circ Y = Y \circ X.$$

That is, $\mathcal{C}$ is an abelian monoid. Conversely, $\mathcal{C}$ determines the monoidal 1-category $B\mathcal{C}$.

Moreover, an E_2-monoidal functor $F : (\mathcal{C}, B\mathcal{C}) \to (\mathcal{D}, B\mathcal{D})$ is a monoidal functor $B\mathcal{C} \to B\mathcal{D}$ which is nothing but a homomorphism of monoids $\mathcal{C} \to \mathcal{D}$. Therefore, $E_2\mathcal{C}\mathrm{at}_0$ is the 1-category of abelian monoids and homomorphisms.

Example

Let $(\mathcal{C}, B\mathcal{C})$ be an E_2-monoidal 1-category. That is, $B\mathcal{C}$ is a monoidal 2-category with a single object $\bullet$ and $\mathcal{C} = \mathrm{Hom}_{B\mathcal{C}}(\bullet, \bullet)$. Forgetting the monoidal structure of $B\mathcal{C}$, we see that $\mathcal{C}$ is a monoidal 1-category with tensor product $\circ$ and tensor unit $\mathrm{Id}_\bullet$.

The monoidal structure of $B\mathcal{C}$ induces natural isomorphisms $X \circ Y \simeq X \otimes Y \simeq Y \circ X$, promoting $\mathcal{C}$ to a braided monoidal 1-category. Conversely, $\mathcal{C}$ determines the monoidal 2-category $B\mathcal{C}$ up to canonical monoidal equivalence. Therefore, $E_2\mathrm{Cat}_1$ is canonically equivalent to the 2-category of braided monoidal 1-categories, braided monoidal functors and braided monoidal natural isomorphisms.

Moreover, for $m \geq 3$, $E_m\mathrm{Cat}_1$ is canonically equivalent to the 2-category of symmetric monoidal 1-categories, symmetric monoidal functors and symmetric monoidal natural isomorphisms.

An E_2-monoidal n-category is also referred to as a braided monoidal n-category.
An E_3-monoidal n-category is also referred to as a syleptic monoidal n-category.

We have an evident forgetful functor $E_{m+1}\mathrm{Cat}_n \to E_m\mathrm{Cat}_n$.

Hypothesis (Delooping hypothesis)

The forgetful functor $E_{m+1}\mathrm{Cat}_n \to E_m\mathrm{Cat}_n$ is an equivalence for $m \geq n+2$.

The $(n+1)$-category $E_\infty\mathrm{Cat}_n$ of symmetric monoidal or E_∞-monoidal n-categories is defined to be $E_m\mathrm{Cat}_n$ for $m \geq n+2$.

For an E_m-monoidal n-category $\mathcal{C}$, we use $\mathcal{C}^{\mathrm{op}k}$ where $k > -m$ to denote the E_m-monoidal n-category obtained by reversing all the k-morphisms, i.e. $B^m(\mathcal{C}^{\mathrm{op}k}) = (B^m\mathcal{C})^{\mathrm{op}(k+m)}$. In particular, $\mathcal{C}^{\mathrm{op}0}$ is denoted by $\mathcal{C}^{\mathrm{rev}}$ and $\mathcal{C}^{\mathrm{op}(-1)}$ is denoted by $\overline{\mathcal{C}}$.

For example, there are three levels of E_m-monoidal 0-categories:

- E_0-monoidal 0-category $=$ set with a distinguished element,
- E_1-monoidal 0-category $=$ monoid,
- E_∞-monoidal 0-category $=$ abelian monoid.

There are four levels of E_m-monoidal 1-categories:

- E_0-monoidal 1-category $=$ 1-category with a distinguished object,
- E_1-monoidal 1-category $=$ monoidal 1-category,
- E_2-monoidal 1-category $=$ braided monoidal 1-category,
- E_∞-monoidal 1-category $=$ symmetric monoidal 1-category.

In general, there are a total of $n+3$ levels of E_m-monoidal n-categories.

The $(n+1)$-categories Cat_n and $E_m\mathrm{Cat}_n$, $0 \leq m \leq \infty$ are symmetric monoidal under Cartesian product. The tensor unit is the trivial n-category.

Note that for $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathrm{Cat}_0$, we have

$$\mathsf{Fun}(\mathcal{A} \times \mathcal{B}, \mathcal{C}) = \mathsf{Fun}(\mathcal{A}, \mathsf{Fun}(\mathcal{B}, \mathcal{C})).$$

In general, for $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathrm{Cat}_n$, we have

$$\mathsf{Fun}(\mathcal{A} \times \mathcal{B}, \mathcal{C}) = \mathsf{Fun}(\mathcal{A}, \mathsf{Fun}(\mathcal{B}, \mathcal{C})).$$

3. Duality

Definition

Let $\mathcal{C}$ be an n-category containing two k-morphisms $f : X \to Y$ and $g : Y \to X$. We say that a f is left dual to g and say that g is right dual to f, if there exist $(k+1)$-morphisms $u : \mathrm{Id}_X \to g \circ f$ and $v : f \circ g \to \mathrm{Id}_Y$ such that the composite $(k+1)$-morphisms

$$f \simeq f \circ \mathrm{Id}_X \xrightarrow{u} f \circ g \circ f \xrightarrow{v} \mathrm{Id}_Y \circ f \simeq f,$$

$$g \simeq \mathrm{Id}_X \circ g \xrightarrow{u} g \circ f \circ g \xrightarrow{v} g \circ \mathrm{Id}_Y \simeq g$$

are equivalent to the identities. We refer to u as the unit map and refer to v as the counit map of the duality. We also denote $g^L = f$ and $f^R = g$.

Remark

If f is inverse to g, then f is both left dual and right dual to g.

In an n-category, two n-morphisms are dual to each other if and only if they are inverse to each other.

Definition

We say that an n-category $\mathcal{C}$ has duals, if every k-morphism has both a left dual and a right dual for $1 \leq k < n$.
We say that an E_0-monoidal n-category $(\mathcal{C}, 1_\mathcal{C})$ has duals, if $\mathcal{C}$ has duals.
We say that an E_m-monoidal n-category $\mathcal{C}$ has duals where $m \geq 1$, if the $(n+1)$-category $B\mathcal{C}$ has duals.

Example

A functor $F : \mathcal{C} \to \mathcal{D}$ is left dual to $G : \mathcal{D} \to \mathcal{C}$ as 1-morphisms in Cat_n if and only if F is left adjoint to G, i.e. there exists a natural equivalence

$$\mathrm{Hom}_\mathcal{D}(F(X), Y) \simeq \mathrm{Hom}_\mathcal{C}(X, G(Y)).$$

4. Module n-categories

Let $\mathcal{C}$ be a monoid and let $F : B\mathcal{C} \to \mathrm{Cat}_0$ be a functor. Unwinding the definition, we see that F consists of a set $F(\bullet) \in \mathrm{Cat}_0$ and a homomorphism of monoids $F : \mathcal{C} \to \mathrm{Fun}(F(\bullet), F(\bullet))$. That is, giving a functor $F : B\mathcal{C} \to \mathrm{Cat}_0$ is equivalent to giving a left $\mathcal{C}$-module $F(\bullet)$.

Let $\xi : F \to G$ be a natural transformation. Unwinding the definition, we see that ξ is a map $\xi_\bullet : F(\bullet) \to G(\bullet)$ rendering the following diagram commutative for all $a \in \mathcal{C}$

$$\begin{tikzcd} F(\bullet) \arrow[r, "\xi_\bullet"] \arrow[d, "F(a)"'] & G(\bullet) \arrow[d, "G(a)"] \\ F(\bullet) \arrow[r, "\xi_\bullet"] & G(\bullet). \end{tikzcd}$$

That is, giving a natural transformation $\xi : F \to G$ is equivalent to giving a left $\mathcal{C}$-module map $\xi_\bullet : F(\bullet) \to G(\bullet)$.

Therefore, $\mathrm{Fun}(B\mathcal{C}, \mathrm{Cat}_0)$ is the 1-category of left $\mathcal{C}$-modules and module maps.

Definition

For a monoidal n-category $\mathcal{C}$, the $(n+1)$-category $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}_n)$ of **left $\mathcal{C}$-modules** is defined to be $\mathrm{Fun}(B\mathcal{C}, \mathrm{Cat}_n)$ and the $(n+1)$-category $\mathrm{RMod}_{\mathcal{C}}(\mathrm{Cat}_n)$ of **right $\mathcal{C}$-modules** is defined to be $\mathrm{Fun}(B\mathcal{C}^{\mathrm{rev}}, \mathrm{Cat}_n)$. We use $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ to denote $\mathrm{Hom}_{\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}_n)}(\mathcal{M}, \mathcal{N})$.

For monoidal n-categories $\mathcal{C}$ and $\mathcal{D}$, the $(n+1)$-category $\mathrm{BMod}_{\mathcal{C}|\mathcal{D}}(\mathrm{Cat}_n)$ of **$\mathcal{C}$-$\mathcal{D}$-bimodules** is defined to be $\mathrm{Fun}(B\mathcal{C}, \mathrm{Fun}(B\mathcal{D}^{\mathrm{rev}}, \mathrm{Cat}_n))$, which is equivalent to $\mathrm{LMod}_{\mathcal{C}\times\mathcal{D}^{\mathrm{rev}}}(\mathrm{Cat}_n)$. We use $\mathrm{Fun}_{\mathcal{C}|\mathcal{D}}(\mathcal{M}, \mathcal{N})$ to denote $\mathrm{Hom}_{\mathrm{BMod}_{\mathcal{C}|\mathcal{D}}(\mathrm{Cat}_n)}(\mathcal{M}, \mathcal{N})$.

5. Additive n-categories

Definition

An additive 0-category is an abelian group $\mathcal{C}$. A functor $F : \mathcal{C} \to \mathcal{D}$ between additive 0-categories is additive if it is a group homomorphism. We use Cat_0^+ to denote the 1-category of additive 0-categories and additive functors, and use $\mathrm{Fun}^+(\mathcal{C}, \mathcal{D})$ to denote $\mathrm{Hom}_{\mathrm{Cat}_0^+}(\mathcal{C}, \mathcal{D})$.

Definition

Let $\mathcal{C}$ be an n-category. We say that an object $X \in \mathcal{C}$ is initial if $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is contractible for all $Y \in \mathcal{C}$. We say that $X \in \mathcal{C}$ is final if $\mathrm{Hom}_{\mathcal{C}}(Y, X)$ is contractible for all $Y \in \mathcal{C}$. We say that $X \in \mathcal{C}$ is zero if X is both initial and final. We say that a 1-morphism of $\mathcal{C}$ is zero if it factors through a zero object.

A zero object of $\mathcal{C}$ is denote by 0. If $\mathcal{C}$ has a zero object, there exists a zero 1-morphism between any two objects of $\mathcal{C}$. A zero 1-morphism is also denote by 0.

Definition

Let $\mathcal{C}$ be an n-category. The product of two objects $X, Y \in \mathcal{C}$ is an object $X \times Y \in \mathcal{C}$ equipped with a natural equivalence $\mathrm{Hom}_{\mathcal{C}}(Z, X \times Y) \simeq \mathrm{Hom}_{\mathcal{C}}(Z, X) \times \mathrm{Hom}_{\mathcal{C}}(Z, Y)$ for $Z \in \mathcal{C}$. The coproduct of two objects $X, Y \in \mathcal{C}$ is an object $X \coprod Y \in \mathcal{C}$ equipped with a natural equivalence $\mathrm{Hom}_{\mathcal{C}}(X \coprod Y, Z) \simeq \mathrm{Hom}_{\mathcal{C}}(X, Z) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z)$ for $Z \in \mathcal{C}$.

Remark

Giving a 1-morphism $Z \to X \times Y$ is equivalent to giving a pair of 1-morphism $Z \to X$ and $Z \to Y$. Giving a 1-morphism $X \coprod Y \to Z$ is equivalent to giving a pair of 1-morphism $X \to Z$ and $Y \to Z$.

There is a canonical 1-morphism $X \to X \times X$ and a canonical 1-morphism $X \coprod X \to X$.

If $\mathcal{C}$ has a zero object 0, then $X \times 0 \simeq X \simeq X \coprod 0$ canonically.

Definition

We say that an n-category $\mathcal{C}$ is quasi-additive if $\mathcal{C}$ has a zero object and finite products as well as finite coproducts such that the canonical 1-morphism $X \coprod Y \to X \times Y$ is invertible for all $X, Y \in \mathcal{C}$. The coproduct $X \coprod Y$ is also denoted by $X \oplus Y$, referred to as the direct sum of X and Y.

Remark

The canonical 1-morphism $X \coprod Y \to X \times Y$ is determined by $X \coprod Y \xrightarrow{\mathrm{Id}_X \coprod 0} X \coprod 0 \simeq X$ and $X \coprod Y \xrightarrow{0 \coprod \mathrm{Id}_Y} 0 \coprod Y \simeq Y$.

If $\mathcal{C}$ is a quasi-additive n-category, then $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ carries a binary operation defined by

$$f + g : X \to X \times X \xrightarrow{f \times g} Y \times Y \simeq Y \coprod Y \to Y$$

for 1-morphisms $f, g : X \to Y$.

Proposition

We have $h \circ (f+g) \simeq h \circ f + h \circ g$ and $(f+g) \circ i \simeq f \circ i + g \circ i$ for $W \xrightarrow{i} X \xrightarrow{f,g} Y \xrightarrow{h} Z$.

Proof.

The first equivalence follows from the commutative diagram:

$$\begin{tikzcd}
X \arrow[r] & X \times X \arrow[r, "f \times g"] \arrow[dr, "h \circ f \times h \circ g"'] & Y \times Y \arrow[d, "h \times h"] & Y \coprod Y \arrow[l, "\sim"'] \arrow[r] \arrow[d, "h \coprod h"] & Y \arrow[d, "h"] \\
& & Z \times Z & Z \coprod Z \arrow[l, "\sim"'] \arrow[r] & Z.
\end{tikzcd}$$

The other one is proved similarly. □

For $n = 1$, the binary operation $(f, g) \mapsto f + g$ endows $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ with the structure of an abelian monoid.

Definition

We say that a 1-category $\mathcal{C}$ is additive if it is quasi-additive and the abelian monoid $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is an abelian group for any objects $X, Y \in \mathcal{C}$.

Definition

By induction on $n \geq 2$, we say that an n-category $\mathcal{C}$ is additive if it is quasi-additive and $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is additive for any objects $X, Y \in \mathcal{C}$ and the canonical 2-morphisms $f \coprod g \to f + g \to f \times g$ are invertible for any 1-morphisms $f, g : X \to Y$.

Remark

The canonical 2-morphism $f + g \to f \times g$ is determined by $f + g \xrightarrow{\mathrm{Id}_f + 0} f + 0 \simeq f$ and $f + g \xrightarrow{0 + \mathrm{Id}_g} 0 + g \simeq g$, and similarly for $f \coprod g \to f + g$. Note that the composition $f \coprod g \to f + g \to f \times g$ agrees with the canonical one. Therefore, in an additive n-category, the binary operation $+$ realizes $\oplus$ on 2-morphisms.

Remark

By definition, if an n-category $\mathcal{C}$ additive, then $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is also additive for $X, Y \in \mathcal{C}$.

Definition

A functor $F : \mathcal{C} \to \mathcal{D}$ between two additive n-categories is additive if F preserves finite products or, equivalently, finite coproducts.
We use Cat_n^+ to denote the subcategory of Cat_n formed by additive n-categories and additive functors, and use $\mathrm{Fun}^+(\mathcal{C}, \mathcal{D})$ to denote $\mathrm{Hom}_{\mathrm{Cat}_n^+}(\mathcal{C}, \mathcal{D})$.

Proposition

If $F : \mathcal{C} \to \mathcal{D}$ is additive, then $F(f+g) \simeq F(f) + F(g)$ for $f, g : X \to Y$. That is, $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is also additive for $X, Y \in \mathcal{C}$.

Proof.

$$\begin{tikzcd}
F(X) \arrow[r] \arrow[dr] & F(X \times X) \arrow[r, "F(f \times g)"] \arrow[d, "\sim"] & F(Y \times Y) \arrow[d, "\sim"] & F(Y \coprod Y) \arrow[l, "\sim"'] \arrow[r] & F(Y). \\
& F(X) \times F(X) \arrow[r, "F(f) \times F(g)"] & F(Y) \times F(Y) & F(Y) \coprod F(Y) \arrow[l, "\sim"'] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

□

Proposition

The $(n+1)$-category Cat_n^+ is additive, and the direct sum $\mathcal{C} \oplus \mathcal{D}$ is the Cartesian product $\mathcal{C} \times \mathcal{D}$.

Proof.

We have $(X, Y) \coprod (X', Y') = (X \coprod X', Y \coprod Y') \simeq (X \times X', Y \times Y') = (X, Y) \times (X', Y')$. Hence $(X, Y) \oplus (X', Y') = (X \oplus X', Y \oplus Y')$. This shows that $\mathcal{C} \times \mathcal{D}$ is additive.

In particular, $(X, 0) \oplus (0, Y) \simeq (X, Y)$. Therefore, $\mathrm{Fun}^+(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}^+(\mathcal{C}, \mathcal{E}) \times \mathrm{Fun}^+(\mathcal{D}, \mathcal{E})$. That is, $\mathcal{C} \times \mathcal{D}$ is the coproduct of $\mathcal{C}$ and $\mathcal{D}$ in Cat_n^+. This shows that Cat_n^+ is quasi-additive.

For additive functors $F, G : \mathcal{C} \to \mathcal{D}$, we have $(F + G)(X) \simeq F(X) \oplus G(X) \simeq (F \oplus G)(X)$. This shows that Cat_n^+ is additive. $\square$

Definition

An additive monoidal n-category is a pair $(\mathcal{C}, B\mathcal{C})$ where $B\mathcal{C}$ is an additive $(n+1)$-category such that all objects are finite direct sums of a single object $\bullet$ and $\mathcal{C} = \mathrm{Hom}_{B\mathcal{C}}(\bullet, \bullet)$. Additive E_m-monoidal n-categories are similarly defined. We use $E_m\mathrm{Cat}_n^+$ to denote the $(n+1)$-category of additive E_m-monoidal n-categories and use $\mathrm{Fun}^{+E_m}(\mathcal{C}, \mathcal{D})$ to denote $\mathrm{Hom}_{E_m\mathrm{Cat}_n^+}(\mathcal{C}, \mathcal{D})$

Definition

For an additive monoidal n-category $\mathcal{C}$, the additive $(n+1)$-category $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}_n^+)$ of additive left $\mathcal{C}$-modules is defined to be $\mathrm{Fun}^+(B\mathcal{C}, \mathrm{Cat}_n^+)$ and we use $\mathrm{Fun}_{\mathcal{C}}^+(\mathcal{M}, \mathcal{N})$ to denote $\mathrm{Hom}_{\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}_n^+)}(\mathcal{M}, \mathcal{N})$. Categories of additive right modules and bimodules are defined similarly.

Example

An additive monoidal 0-category $\mathcal{C}$ is a ring and $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}_0^+)$ is the additive 1-category of left $\mathcal{C}$-modules and module maps. An additive symmetric monoidal 0-category $\mathcal{C}$ is a commutative ring.

For example, $B\mathbb{C}$ is equivalent to the symmetric monoidal 1-category of finite-dimensional vector spaces over $\mathbb{C}$.

6. Linear n-categories

Let R be a commutative ring (for example, $\mathbb{Z}$ or $\mathbb{C}$) and view R as an additive symmetric monoidal 0-category.

Definition

The $(n+1)$-category Cat_n^R of R-linear n-categories is defined to be $\mathrm{LMod}_{B^n R}(\mathrm{Cat}_n^+) = \mathrm{Fun}^+(B^{n+1}R, \mathrm{Cat}_n^+)$. We use $\mathrm{Fun}_R(\mathcal{C}, \mathcal{D})$ to denote $\mathrm{Hom}_{\mathrm{Cat}_n^R}(\mathcal{C}, \mathcal{D})$, the R-linear n-category formed by R-linear functors and R-linear (higher) natural transformations.

Example

(1) $\mathrm{Cat}_0^R = \mathrm{Fun}^+(BR, \mathrm{Cat}_0^+)$ is the 1-category of R-modules.

(2) $\mathrm{Cat}_n^{\mathbb{Z}} = \mathrm{Fun}^+(B^{n+1}\mathbb{Z}, \mathrm{Cat}_n^+) \simeq \mathrm{Cat}_n^+$.

Example

An R-linear 1-category is by definition an additive 1-category $\mathcal{C}$ equipped with an additive monoidal functor $F : BR \to \mathrm{Fun}^{+}(\mathcal{C}, \mathcal{C})$. Note that $F(\bullet) = \mathrm{Id}_{\mathcal{C}}$ and we have a natural transformation $F(\lambda) : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ for every $\lambda \in R$ such that $F(\lambda\mu) = F(\lambda) \circ F(\mu)$. We define $\lambda f = F(\lambda)_Y \circ f = f \circ F(\lambda)_X$ for 1-morphism $f : X \to Y$ of $\mathcal{C}$. Then $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is promoted to an R-module. Moreover, $(\lambda g) \circ f = \lambda(g \circ f) = g \circ (\lambda f)$. Therefore, $\mathcal{C}$ is promoted to a usual R-linear 1-category.

Definition

An R-linear monoidal n-category is a pair $(\mathcal{C}, B\mathcal{C})$ where $B\mathcal{C}$ is an R-linear $(n+1)$-category such that all objects are finite direct sums of a single object $\bullet$ and $\mathcal{C} = \mathrm{Hom}_{B\mathcal{C}}(\bullet, \bullet)$. An R-linear E_m-monoidal n-category is similarly defined. We use $E_m\mathrm{Cat}_n^R$ to denote the $(n+1)$-category of R-linear E_m-monoidal n-categories.

Definition

For an R-linear monoidal n-category $\mathcal{C}$, the R-linear $(n+1)$-category $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}_n^R)$ of R-linear left $\mathcal{C}$-modules is defined to be $\mathrm{Fun}_R(B\mathcal{C}, \mathrm{Cat}_n^R)$ and we use $\mathrm{Fun}_{\mathcal{C}}^R(\mathcal{M}, \mathcal{N})$ to denote $\mathrm{Hom}_{\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}_n^R)}(\mathcal{M}, \mathcal{N})$. Categories of R-linear right modules and bimodules are defined similarly.

Example

An R-linear monoidal 0-category $\mathcal{C}$ is a R-algebra and $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}_0^+)$ is the R-linear 1-category of left $\mathcal{C}$-modules and module maps. An R-linear symmetric monoidal 0-category $\mathcal{C}$ is a commutative R-algebra.