Lecture 4 — 23 March 2022

P $V \times V$ matrix. Stochastic matrix associated to random walk,

Id $V \times V$ identity matrix $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

$(Id - P)u = f$ discrete Poisson's eq.

If $f = 0$, $(Id - P)u = 0$ discrete Laplace's eq.

We call $Id - P$ discrete Laplace operator.

(Sometimes: $(\Delta u)_x := \sum_{y \in V(x)} c_{xy} (u_y - u_x)$

$\Delta = -C(Id - P)$

C $V \times V$ matrix $\times$ diagonal

$$\begin{array}{c} \\ \\ \\ \end{array}\begin{pmatrix} & x & \\ & \vdots & \\ \cdots & \sum_{y \in V(x)} c(x,y) & \end{pmatrix}$$

$G(x,y)$ Green's function

Green's operator $\begin{cases} G: & \mathbb{R}^{|V|} \longrightarrow \mathbb{R}^{|V|} \\ & f \longmapsto G(f) \end{cases}$ where $(G(f))_x = \sum_{y \in V} G(x,y) f(y)$.

Important: (1). $G(f)$ might not be well defined

(2). $\underbrace{G = (Id - P)^{-1}}$ it only makes sense if we specify which space we are considering.

Ex. Not true if we consider $\mathbb{R}^{|V|}$

Check: $u \equiv 1$. Then we have $(Id - P)u = 0$

$$(G(u))_x = +\infty \quad \forall x \in V.$$

$$(G(u))_x = \sum_y G(x,y) \cdot 1$$

$=$ expected number of visits to some vertex for paths starting from x.

$= +\infty$.

However: If we restrict to $W \subset \mathbb{R}^{|V|}$ which have finite support.

$\Rightarrow \quad \ker((Id - P)|_W) = \{0\}$.

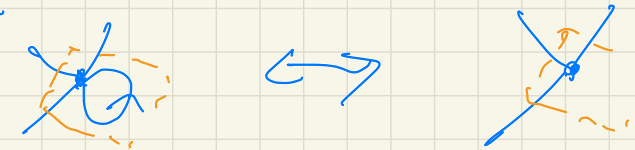
$G|_W$ is always well defined

$$\Rightarrow \quad G|_W = ((Id - P)|_W)^{-1}$$

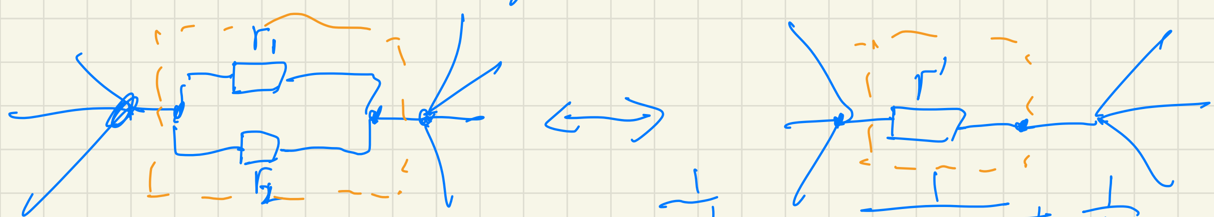
(Keep in mind: We don't consider $\mathbb{R}^{|V|}$
but we care about those with finite energy,
which form a Hilbert space)

Remark: (Γ, r). There are local move on (Γ, r) that preserves voltage and current outside ⬚

① No self-loop ($i(x,x) = -i(x,x) = 0$)

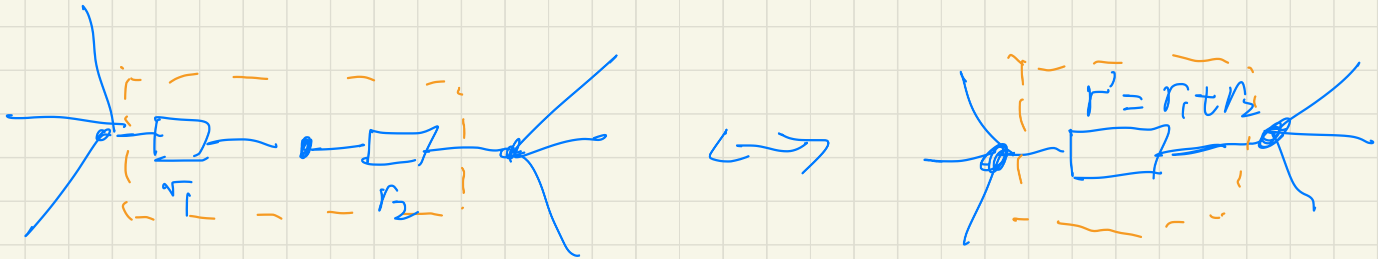


② No multiple edge

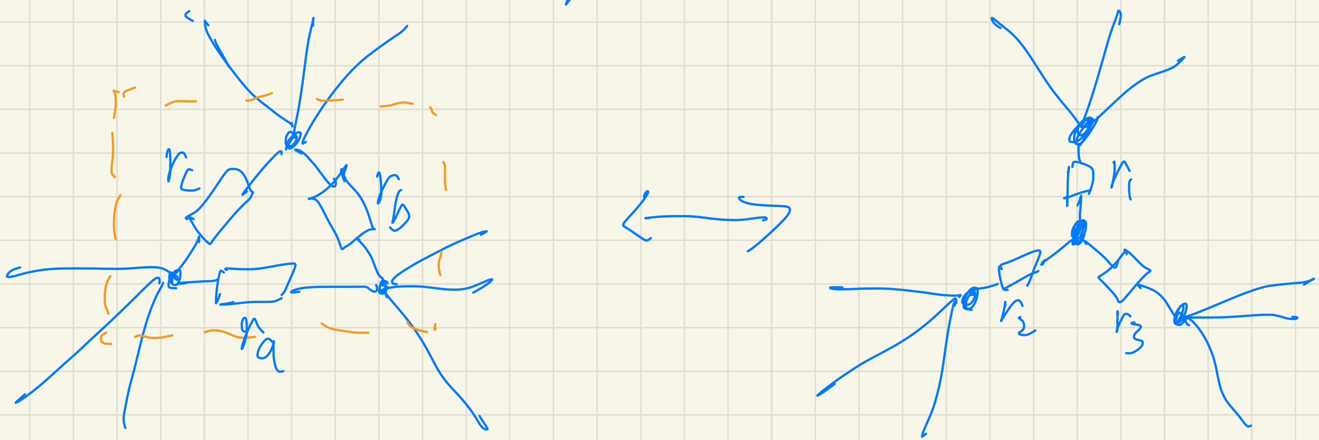


$$\frac{1}{r'} = \frac{1}{r_1} + \frac{1}{r_2}$$

③, No vertex of degree 2,



④, Star-triangle move / Y-Δ move / Yang-Baxter relation.



$$r_1 = \frac{r_b r_c}{r_a + r_b + r_c}, \quad r_2 = \frac{r_a r_c}{r_a + r_b + r_c}, \quad r_3 = \frac{r_a r_b}{r_a r_b r_c}$$

Harmonic functions

① $V' \subset V$, $V' \neq \emptyset$. We say u harmonic on V' if $\left((Id-P)u\right)_x = 0 \quad \forall x \in V'$

If $V' = V$, we say u harmonic on (Γ, r)

② $u: V \to \mathbb{R}$ superharmonic if $(Id - P)u \geq 0$

u subharmonic if $(Id - P)u \leq 0$

Ex. If (V, r) transient,

For each y, $G(\cdot, y) : V \to \mathbb{R}$ is superharmonic

and harmonic on $(V - \{y\})$

Check: $$PG = P \sum_{n\geq 0}^{\infty} P^n = G - Id$$

$$\Rightarrow (Id - P)\, G = Id$$

$$\Rightarrow (Id - P) \begin{pmatrix} | \\ G(\cdot, y) \\ | \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{pmatrix} \leftarrow y\text{-th row.}$$

y-th column $\qquad$ y-th column of Id

$$\Rightarrow \quad \Big((Id - P)\, G(\cdot\, , y) \Big)_x = \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y \end{cases}$$

For $y \in V$,

$\Rightarrow$ $G(\cdot, y)$ is non-negative superharmonic function.

<u>Strong Min principle</u>

If f superharmonic and $\exists\, x \in V$ s.t.

$f(x) = \min\limits_V f$, then f is constant.

Pf $\quad (Id - P) f \geq 0$

$(\ast) \quad f(x) \geq \sum_y p(x,y) f(y)$

$\geq \sum_y p(x,y) \sum_z p(y,z) f(z)$

$= \sum_z \left(\sum_y p(x,y) p(y,z) \right) f(z)$

$= \sum_z p^{(2)}(x,z) f(z) \geq \ldots$

Suppose $f(x) = \min_V f \Rightarrow f(y) \geq f(x) \quad \forall\, y \in V,$

$(\ast) \quad f(x) \geq \sum_y p^n(x,y) f(y) \geq \sum_y p^n(x,y) f(x)$

$$= f(x)$$

$\Rightarrow$ no strict inequality

$\Rightarrow$ $f(x) = f(y)$ as long as $P^n(x,y) > 0$ for some n.

$\Rightarrow$ f is constant.

Thm (Γ, r) is recurrent if and if all non-negative superharmonic functions are constant.

(Recall: (Γ, r) transient $\Rightarrow$ $G(\cdot, y)$ non-negative superharmonic, nonstant.)

Pf: $(\Leftarrow)$ (Γ, r) transient $\Rightarrow$ $\exists$ non-constant non-negative superharmonic function.

($\Leftarrow$) Assume (Γ, r) recurrent.

Let $f \geq 0$ superharmonic

$$g := (\mathrm{Id} - P) f \geq 0$$

Claim $\vdash$: $g \equiv 0$.

Suppose not. $g(y) > 0$ for some $y \in V$.

Pick any $x \in V$.

$$(P^0 + P^1 + P^2 + \dots + P^n) g = \sum_{k=0}^{n} P^k (\mathrm{Id} - P) f$$

$$= (Id - P^{n+1}) f$$

~~Consider the x-th row~~

$$\sum_{k=0}^{n} P^{(k)}(x,y)\, g(y) \le \sum_{y \in V} \sum_{k=0}^{n} P^{(k)}(x,y)\, g(y)$$

$$= x\text{-th row of } (P^0 + P^1 + \dots + P^n) g$$

$$= x\text{-th row of } (Id - P^{n+1}) f$$

$$= f(x) - \sum_{y \in V} P^{(n+1)}(x,y)\, f(y)$$

$$\leq f(x)$$

$$\Rightarrow \quad \sum_{k=0}^{n} p^{(k)}(x,y) \leq \frac{f(x)}{g(y)}, \quad \text{for all } n.$$

$\Rightarrow$ Take $n \to \infty$

$$G(x,y) \leq \frac{f(x)}{g(y)} \quad \text{is finite}$$

$\Rightarrow$ contradiction since $G(x,y) = \infty$ for (Γ, r) recurrent.

$$\Rightarrow \quad g(y) = 0 \qquad \forall\, y \in V$$

$$\Rightarrow \quad (\mathrm{Id} - P) f = g = 0,$$

$$\Rightarrow \quad f \text{ harmonic}$$

(Lemma: (Γ, r) recurrent. Then all nonnegative superharmonic functions are harmonic)

Let $f \geq 0$ superharmonic.

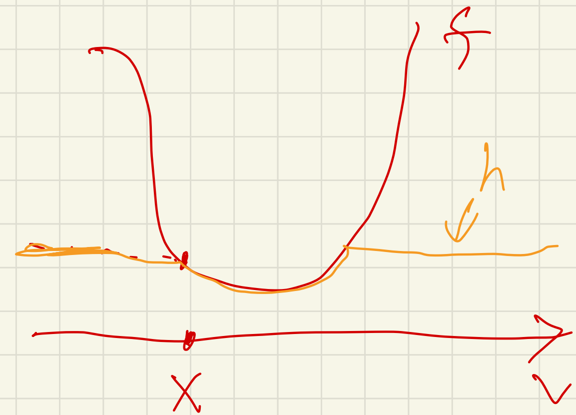
Pick $x \in X$. Set $M := f(x)$,

$$h(y) := \min\{M, f(y)\} \quad \text{for all } y \in V$$

$\Rightarrow \quad h : V \to \mathbb{R}$.

Check: ① h takes max value at x. $h(x) = M$

$f(x) = M$



② $h \geq 0$ superharmonic. $\Rightarrow$ harmonic

Pf: $\forall\, x \in V$.

$$\sum_y p(x,y)\, h(y) \leq \sum_y p(x,y)\, f(y)$$

$$Ph \leq Pf$$

$$(Id-P)\,h \ge (Id-P)\,f \ge 0 \quad \#$$

①+② $\Rightarrow$ h is constant by Max principle.

$\Rightarrow$ Choose M big enough,

we deduce f is constant $\#$.

$G(\cdot\,;y)$ y-th column of $G \in \mathbb{R}^M$

We will consider:

- non-negative superharmonic
- harmonic Dirichlet function
- bounded harmonic function
- bounded harmonic Dirichlet function.
- non-negative harmonic.