Examples of monoidal categories

Last week, we finished by the example Vec_G

Today, we will start by defining Vec_G^ω (like Vec_G but with a deformation ω)

Consider the simple objects of Vec_G: one-dim. vector space δ_g with $(\delta_g)_h = \delta_{g,h} \cdot \delta_g$
($1 = \delta_1 = k$)

← Kronecker symbol

Then: $\delta_g \otimes \delta_h = \delta_{g \cdot h}$

And: $a_{\delta_g, \delta_h, \delta_m} = id_{\delta_{ghm}} : \underbrace{(\delta_g \otimes \delta_h) \otimes \delta_m}_{\delta_{ghm}} \longrightarrow \underbrace{\delta_g \otimes (\delta_h \otimes \delta_m)}_{\delta_{ghm}}$ (trivial)

We will make a modification on the associativity constraint a, to define Vec_G^ω:

<u>Now:</u> $a_{\delta_g, \delta_h, \delta_m} = \omega(g,h,m)\, \mathrm{id}_{\delta_{ghm}}$.

with $\omega : G \times G \times G \to \mathbb{R}^*$ such that (Pentagon Axiom)

$$\omega(g_1 g_2, g_3, g_4)\,\omega(g_1, g_2, g_3 g_4) = \omega(g_1, g_2, g_3)\,\omega(g_1, g_2 g_3, g_4)\,\omega(g_2, g_3, g_4)$$

(such an ω is called a 3-cocycle).

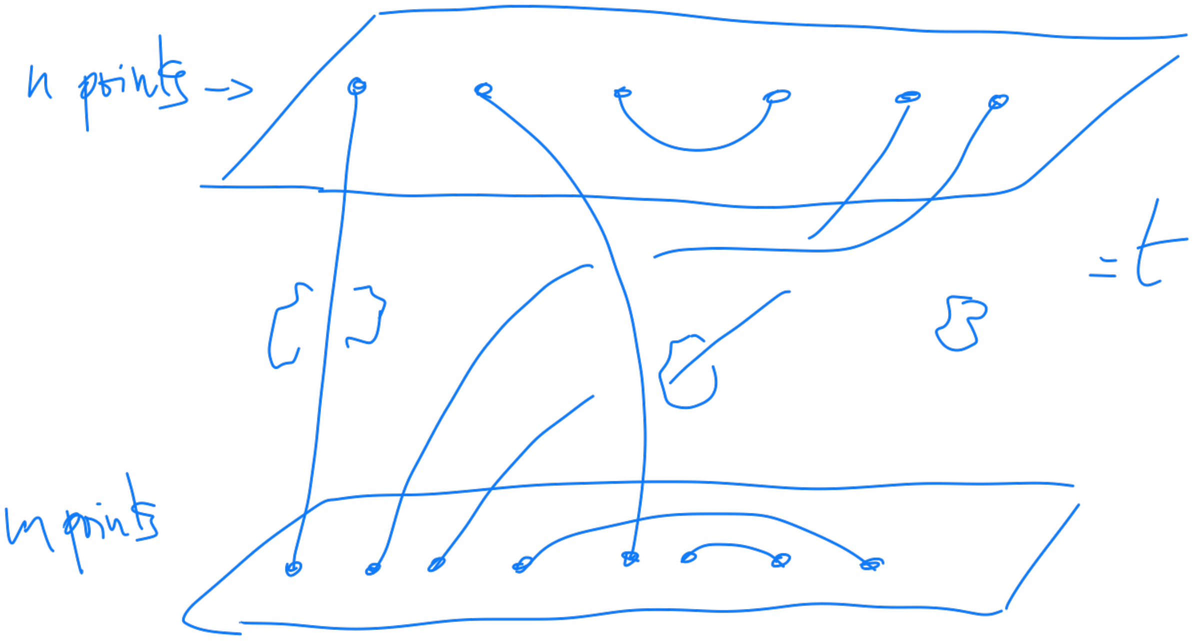
Now, an example where the objects are NOT sets:

$\boxed{\text{Category of tangles } \mathcal{C}}$.

- objects: numbers $n \in \mathbb{N}_{\geq 0}$
- $\otimes$: $n \otimes m = n + m$
- unit: 0

$\left\{ a, l \right.$: obvious

- morphisms : $\hom_{\mathcal{E}}(n,m) = T(n,m)$ set of tangles as below (up to isotopy)

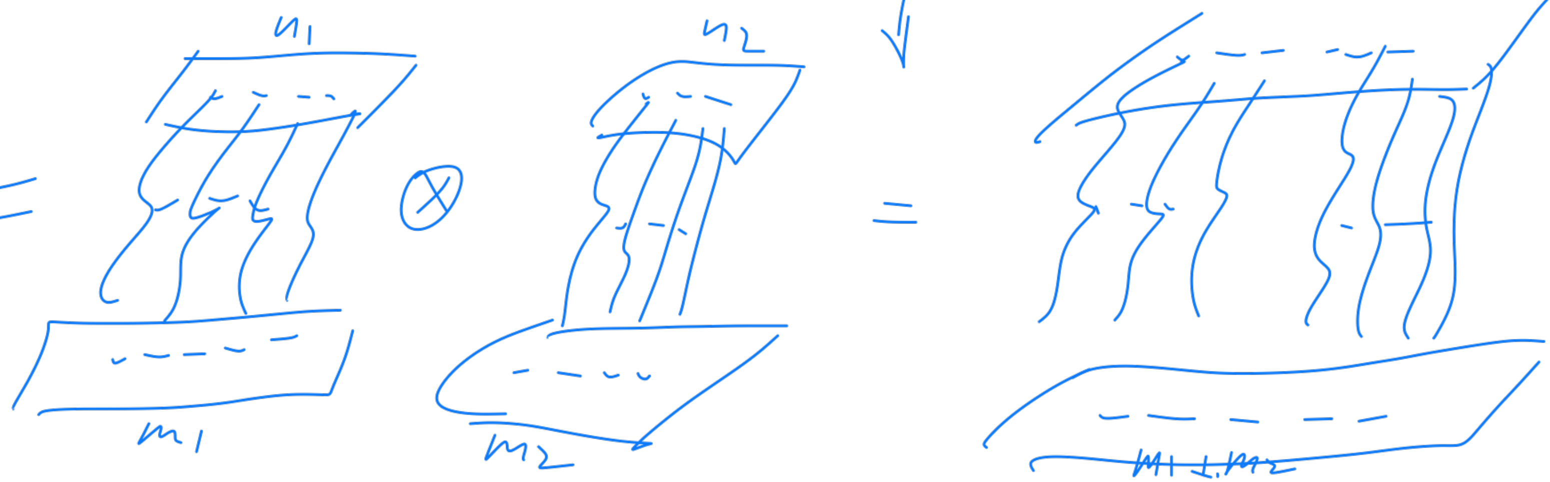


(here) $= t \in T(6,8)$

$\otimes$ of morphisms:

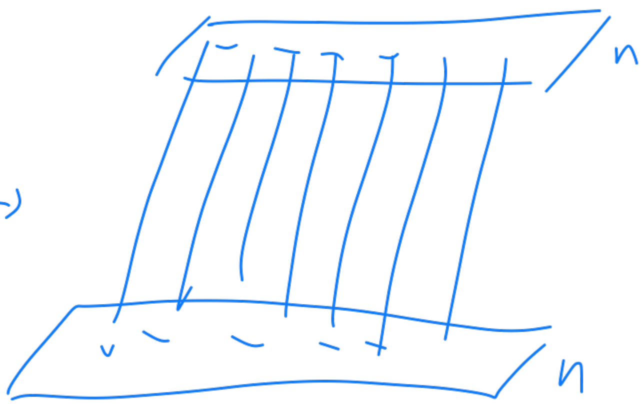
horizontal (concatenation)

$t_1 \otimes t_2 =$



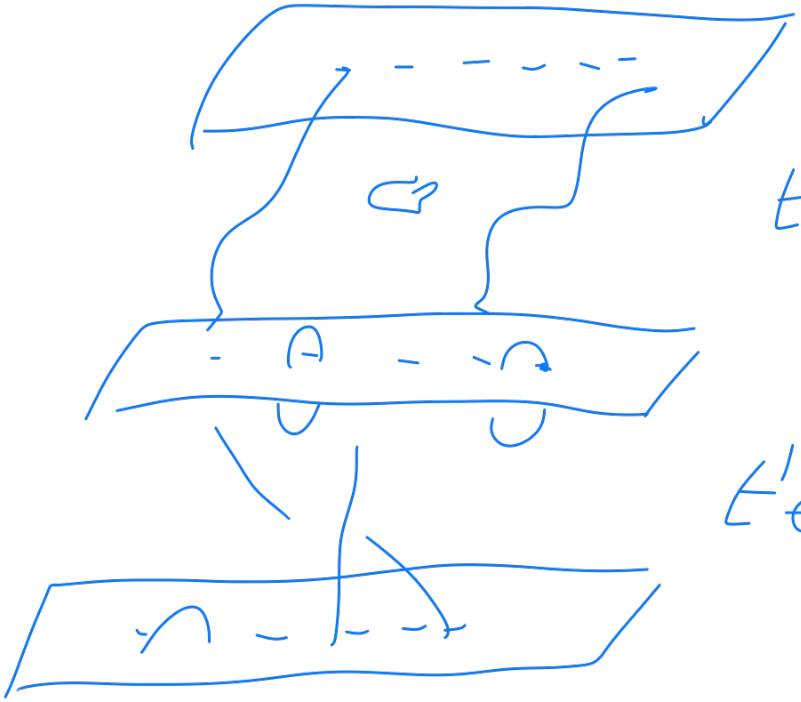
- identities:

straight lines $\longrightarrow$



$\in T(n,n)$

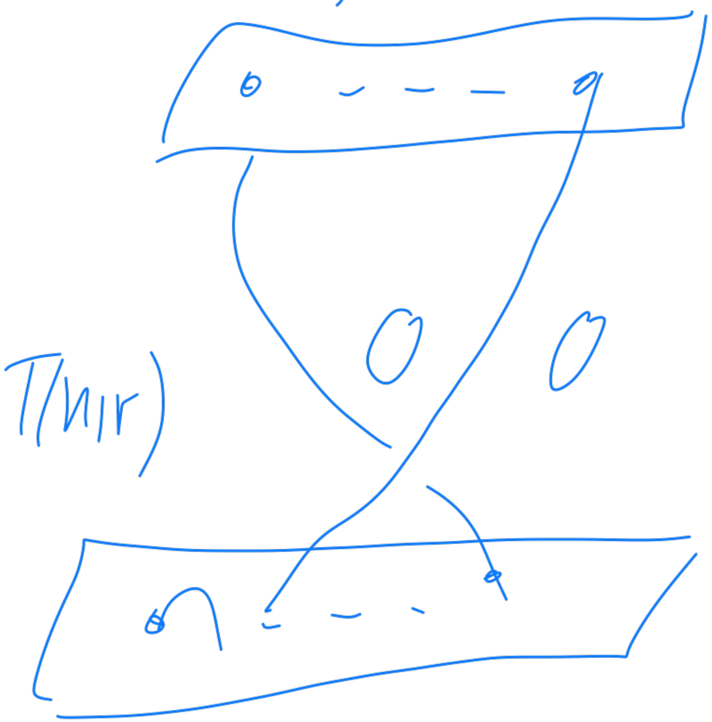
- composition of tangles (vertical concatenation):



$t \in T(n,m)$

$t' \in T(m,r)$

$\Bigg\}\ t' \circ t \in T(n,r)$



--- (Summary of last semester course 5) ---

Monoidal functors: between two monoidal categories $(\mathcal{C}, \otimes, 1, a, \iota)$ and $(\mathcal{C}', \otimes', 1', a', \iota')$

is a couple (F, J), where $F : \mathcal{C} \to \mathcal{C}'$ functor and J is a natural isomorphism:

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{C} \to \mathcal{C}' & \xrightarrow{J} & \mathcal{C} \times \mathcal{C} \to \mathcal{C}' \\ (X, Y) \mapsto F(X) \otimes' F(Y) & & (X, Y) \mapsto F(X \otimes Y) \end{array}$$

[then: $\forall X, Y \in \mathcal{C}$, $J_{X,Y} : F(X) \otimes' F(Y) \xrightarrow{\sim} F(X \otimes Y)$

+ commuting diagrams of natural transformation)] satisfying the following two axioms (made to avoid any ambiguity):

(1) <u>Unit Axiom</u>: $F(1) \sim 1'$, i.e. $\exists\, \varphi: 1' \xrightarrow{\sim} F(1)$ isom.

(2) <u>Monoidal structure axiom</u>: $\forall X, Y, Z \in \mathcal{C}$, the following commutes:

$$\begin{tikzcd}
(F(X) \otimes' F(Y)) \otimes' F(Z) \arrow[r, "a'_{F(X), F(Y), F(Z)}"] \arrow[d, "J_{X,Y} \otimes \mathrm{id}_{F(Z)}"'] & F(X) \otimes' (F(Y) \otimes' F(Z)) \arrow[d, "\mathrm{id}_{F(X)} \otimes J_{Y,Z}"] \\
F(X \otimes Y) \otimes' F(Z) \arrow[d, "J_{X \otimes Y, Z}"'] & F(X) \otimes' F(Y \otimes Z) \arrow[d, "J_{X, Y \otimes Z}"] \\
F((X \otimes Y) \otimes Z) \arrow[r, "F(a_{X,Y,Z})"'] & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

Rk: A functor can have zero or one or several non-equivalent monoidal structure (see examples later).

To solve extra-ambiguity involving the unit object:

Theorem: There is a canonical isomorphism $\varphi : 1' \to F(1)$ s.t.

$$\begin{tikzcd}
1' \otimes' F(1) \arrow[r, "l'_{F(1)}"] \arrow[d, "\varphi \otimes' id_{F(1)}"'] & F(1) \arrow[d, "F(l_1)^{-1}"] \\
F(1) \otimes F(1) \arrow[r, "J_{1,1}"] & F(1,1)
\end{tikzcd}$$

commutes.

Rk: Above commuting diagram can be generalized where $1 \rightsquigarrow X$ and there is also an analogous with r (instead of l). In other words, $\forall X \in \mathcal{C}$, the following diagrams commute:

$$\begin{tikzcd} 1'\otimes F(X) \arrow[r, "l'_{F(X)}"] \arrow[d, "\varphi\otimes' id_{F(X)}"'] & F(X) \arrow[d, "F(l_X)^{-1}"] \\ F(1)\otimes' F(X) \arrow[r, "J_{1,X}"'] & F(1\otimes X) \end{tikzcd}$$

and

$$\begin{tikzcd} F(X)\otimes' 1' \arrow[r, "r'_{F(X)}"] \arrow[d, "id_{F(X)}\otimes' \varphi"'] & F(X) \arrow[d, "F(r_X)^{-1}"] \\ F(X)\otimes' F(1) \arrow[r, "J_{X,1}"'] & F(X\otimes 1) \end{tikzcd}$$

Rk: A monoidal functor can alternatively be defined as a triple (F, J, φ) satisfying:
- monoidal structure axiom,
- two commuting diagrams above.

But, the def we gave (following the book) is simpler, as don't assume the existence of φ (it is just proved to exist)

Convention: $1'$ and $F(1)$ can safely be identified.

Then $F(1)=1'$, $\varphi=id_1$, $J_{1,X}=J_{X,1}=id_X$.

Natural transformation of monoidal functors:

$$\begin{tikzcd}[column sep=huge] (\mathcal{C}, \otimes, 1, a, \iota) \arrow[r, bend left=50, "{(F^1, J^1)}"{name=U}] \arrow[r, bend right=50, "{(F^2, J^2)}"'{name=D}] & (\mathcal{C}', \otimes', 1', a', \iota') \arrow[Rightarrow, from=U, to=D, "\eta"] \end{tikzcd}$$

with:

- natural transformation $\eta : F^1 \to F^2$
- isomorphism: $F^1(1) \xrightarrow{\eta_1} F^2(1)$

$$\begin{matrix} F^1(1) & \xrightarrow{\eta_1} & F^2(1) \\ \| & & \| \\ 1' & & 1' \end{matrix} \quad \leftarrow \text{convention}$$

- the following diagram commutes $\forall X, Y \in \mathcal{C}$:

$$\begin{tikzcd}
F^1(X) \otimes' F^1(Y) \arrow[r, "J^1_{X,Y}"] \arrow[d, "\eta_X \otimes' \eta_Y"'] & F^1(X \otimes Y) \arrow[d, "\eta_{X \otimes Y}"] \\
F^2(X) \otimes' F^2(Y) \arrow[r, "J^2_{X,Y}"] & F^2(X \otimes Y)
\end{tikzcd}$$

--- (Summary of last semester course 6) ---

Examples of monoidal functors and natural transformation between them.

Let $f : H \to G$ be a (finite) group morphism. The following f^* and f_* are monoidal functors:

- The restriction of a rep. V of G to $f(H) < G$ provides a rep of H. Notation: $f^* : \mathrm{Rep}(G) \to \mathrm{Rep}(H)$

- Let W be a H-graded vector space : $W = \bigoplus_{h \in H} W_h$.

It is also G-graded as : $W = \bigoplus_{g \in G} \left(\bigoplus_{h \in f^{-1}(\{g\})} W_h \right)$

Notation: $f_* : Vec_H \to Vec_G$

- Let S be a monoid, consider $\mathcal{C} = Vec_S$

the identity functor $id_{\mathcal{C}}$ is a monoidal functor (where $J_{X,Y} = id_{X \otimes Y}$)

A natural transformation of monoidal functor $\eta : id_{\mathcal{C}} \to id_{\mathcal{C}}$

is completely characterized by a map $\tilde{\eta} : S \to \Bbbk$ (Schur's lemma)

(because, for $s \in S$, δ_s simple, $\eta_{\delta_s} \in hom_{\mathcal{C}}(\delta_s, \delta_s) = \Bbbk \, id_{\delta_s}$

s.t. $\tilde{\eta}$ is a morphism of monoid.

- Monoidal functors between $\mathrm{Vec}_{G_1}^{\omega_1}$ and $\mathrm{Vec}_{G_2}^{\omega_2}$.

It is completely characterized by two maps f, ψ, so that such functor is denoted $F_{f,\psi}$, where:

$$f: G_1 \to G_2 \quad \text{group morphism}$$
$$\psi: G_1 \times G_2 \to \mathbb{k}^* \quad \text{maps} \quad \text{s.t.}$$

$(*)$ $J_{g,h} = \psi(g,h)\, \mathrm{id}_{\delta_{f(gh)}}$ $\qquad \boxed{\forall g,h,\ell \in G_1}$

$(**)$ $\psi(g,h\ell)\cdot\psi(h,\ell)\,\omega_2(f(g),f(h),f(\ell)) = \omega_1(g,h,\ell)\psi(gh,\ell)\psi(g,h)$

.... (Summary of last semester session 7) ... (before ~~course~~ session)

The natural monoidal transformations $\eta : F_{f,\mu} \to F_{f',\mu'}$

First: if such η $(\neq 0)$ exists then $f = f'$ and η_g iso. i.e. $\eta_g = \tilde{\eta}(g)\, id_{f(g)}$

and then, it is completely characterized by $\tilde{\eta} : G \to k^*$

$$\text{s.t.} \quad \tilde{\eta}_{gh} \cdot \mu(g,h) = \mu'(g,h)\,(\tilde{\eta}_g \times \tilde{\eta}_h)$$

Rk: By above result, the 3-cocycle ω for Vec_G^ω can be taken "normalized" (w/o (without) loss of generality) (up to equivalence) :

$$\omega(g,1,h) = 1 \quad \forall g,h \in G$$

If $G = \langle g \mid g^n = e \rangle$ cyclic group of order n, then $\exists\, s \in \mathbb{N}$, $0 \leq s < n$

s.t. $\omega = \omega_s$ with $\omega_s(g^a, g^b, g^c) = \exp\left(\frac{2\pi i}{n} \cdot s \cdot a \cdot \left\lfloor \frac{b+c}{n} \right\rfloor\right)$

Next time: we will start by defining the notion of Group action on categories and equivariantization.

(book, subsection 2.7)