Lecture 2 — 16 March 2022

Goal: Derive Laplace's eq. from electric network.

Given $u : V \to \mathbb{R}$, $\Delta u : V \to \mathbb{R}$ s.t.

$$(\Delta u)_x = \sum_{y \in V(x)} c_{xy} (u_y - u_x)$$

Def Γ graph $\Gamma = (V, Y)$

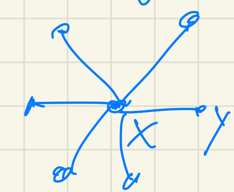
V vertices $\ni x, y$, $Y \subset V \times V$ edges (unoriented)

i.e. $(x,y) \in Y \iff (y,x) \in Y$

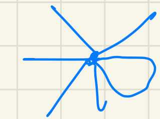
We say $x \sim y$ if $(x,y) \in Y$.

$$V(x) := \{ y \in V \mid x \sim y \}.$$

$\deg(x) = 6$



$x \in V(x) \iff \exists$ self loop



$\deg(x) =$ Cardinality of $V(x)$

Γ is locally finite if $\deg(x) < \infty$ for $x \in V$

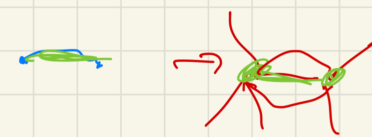
Def. $\Gamma_1 = (V_1, Y_1)$, $\Gamma_2 = (V_2, Y_2)$

ϕ is a (morphism) map from Γ_1 to Γ_2 if

$\phi: V_1 \to V_2$

$Y_1 \to Y_2$ s.t.

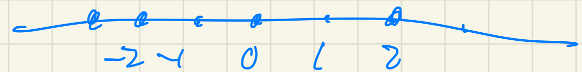
Γ_1



$x \sim y \in V_1 \implies \phi(x) \sim \phi(y) \in V_2$

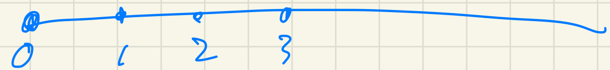
If ϕ has inverse, we say ϕ is isomorphism.

Eg. $\Gamma = \mathbb{Z} = \mathbb{Z}^1$



$\Gamma = \mathbb{Z}^n$

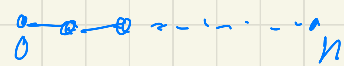
$\mathbb{Z}_+$



Def (1) An infinite path in Γ is a subgraph isomorphic to $\mathbb{Z}$

(2) One-ended infinite path in Γ is a subgraph isomorphic to $\mathbb{Z}_+$.

(3), path of length n is a subgraph isomorphic to



Def Γ is path connected if for any two $x, y \in V$, $\exists$ a path connecting x to y.

Assumption 1

Remark Γ is assumed to be path connected.

Def Combinatorial distance $d(x,y)$ is the minimum n s.t. $\exists$ path of length n connecting to $x, y \in V$.

Ass 2.

Remark We assume Γ is countable ($\Leftrightarrow$ V is countable.)

Ex. locally finite $\Rightarrow$ Γ is countable.

Assumption 3.

Def. (Γ, r) $r : E \to \mathbb{R}_{>0}$ s.t. $r(x,y) = r(y,x)$

and $c(x) := \sum_{y \in V(x)} \frac{1}{r(x,y)} < \infty$. for all $x \in V$.

$c(x)$ is called conductance at $x \in V$.

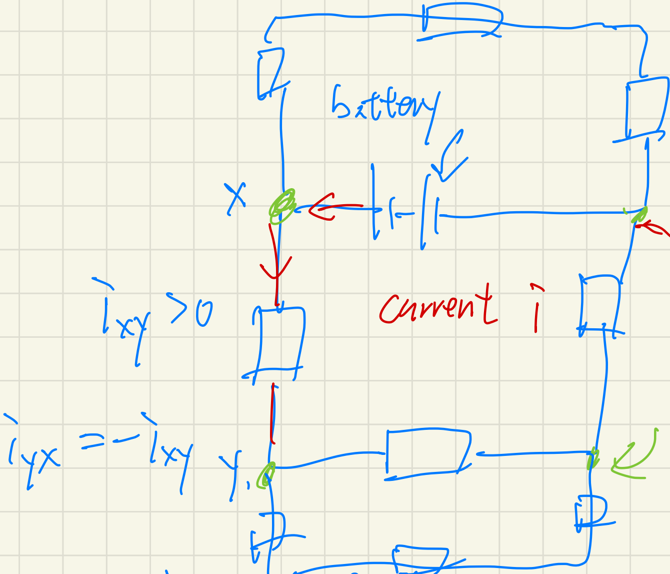
Recall : r resistance 电阻

$c(x,y) := \frac{1}{r(x,y)}$ conductance

$$p(x,y) \sim \frac{c(x,y)}{\sum_{y \in V(x)} c(x,y)}$$

$\deg(x) < \infty \Rightarrow c(x) < \infty$

electric network



resistance r

grounded

Voltage at nodes = vertices.

Kirchoff's equation

is no external source is present.

node: Σ outgoing currents = Σ incoming currents (Conservation of electrons)

closed loop: sum of voltage around closed loop is zero if no battery is present.

$r(x,y)\, i(x,y)$ = Voltage drop from x to y.

① Represent currents as 1-chains.

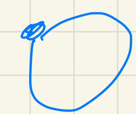
Def. 1-chain on Γ is real value function over oriented edges s.t.

$$i(x,y) = -i(y,x) \in \mathbb{R} \quad \forall\, x \sim y.$$

i is finite if it is non-zero over finitely many edges.

Note: $i(x,x) = 0$

$\Rightarrow$ self-loops are redundant



②. Represent external current as 0-chain.

Def. 0-chain is $j : V \to \mathbb{R}$.

j is finitely supported if nonzero over finitely many edges.

Def. Let C vector space of all 1-chains

s.t. $\sum_{y \in V(x)} |i(x,y)| < \infty \quad \forall x \in V$.

Def. Given 1-chain i, boundary ∂i is 0-chain

defined by $(\partial i)_x = \sum_{y \in V(x)} i(x,y)$ i.e. $\partial i : V \to \mathbb{R}$

sum of currents at vertex i.

Def. Given 0-chain j, boundary $\partial j \in \mathbb{R}$ defined by

$$\partial j := \sum_{x \in V} j(x)$$

Ex: $\partial \partial i = 0$ for any 1-chain i.

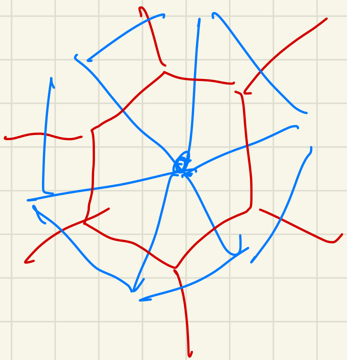
Def 1-chain z is called a cycle if

$$(\partial z)_x = 0 \quad \forall x \in V.$$

Remark : 1-chain $\sim$ differential 1-form on Γ^*

$i(x,y) = -i(y,x) \sim w(x) = -w(-x)$ on $\mathbb{R}$,

image Γ is planar



$\partial i = \sum_{y \in V(x)} i(x,y) \sim dw = 0 \Leftrightarrow \int_\gamma w = 0$

exterior derivative

where γ is contractable closed path.

Γ^* dual graph

1-cochain $\sim$ 1-form on Γ

Def (1) A linear functional on vector space of all finite 1-chains is called a 1-cochain,

(2) . . . .

. . finite 0-chains is . . . . 0-cochain.

$$\delta_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x. \end{cases}$$ is 0-chain

0-chain : $$j = \sum_{x \in V} j(x)\, \delta_x$$

χ_x is functional over 0-chain

$$\chi_x(\delta_{\tilde{x}}) = \begin{cases} 1 & \text{if } x = \tilde{x} \\ 0 & \text{if } x \neq \tilde{x} \end{cases}$$

0 - cochain: $u(x) = \sum_{x \in V} u(x) \chi_x$

$$\langle u, j \rangle = \sum_{x \in V} u(x) j(x)$$

1-chain and 1-cochain: $\varepsilon = \{X, \bar{X}\}$

Fix an orientation for edge. $X \subset Y$

$(x,y) \in X \Rightarrow (y,x) \notin X$.

Given $B, D \in X$, $\delta_B(D) = \begin{cases} 1 & \text{if } B = D \\ 0 & \text{if } B \neq D \end{cases}$

1-chain $i = \sum_{B \in X} i(B) \delta_B$

$$\chi_B(\delta_D) = \begin{cases} 1 & \text{if } B = D \\ 0 & \text{if } B \neq D. \end{cases}$$

1-cochain $\quad E = \sum_{B \in X} E(B)\,\chi_B$

$$\langle E, i \rangle = \sum_{B \in X} E(B)\, i(B)$$

Prop: Given 0-cochain u, coboundary $\partial^* u$ of u is the unique 1-cochain s.t.

$$\langle \partial^* u, k \rangle = \langle u, \partial k \rangle$$

for all finite 1-chain k.

Ex.

$u: V \to \mathbb{R}$.

$$\partial^* u(\delta_{xy}) = u_y - u_x$$

$$\| $$

$$\partial^* u(xy).$$

Main message:

summation at vertices ∂ $\sim$ boundary

difference along edges ∂^* coboundary.