K3 surfaces. $(\mathbb{P}^n = \mathbb{C}P^n)$

① quartic surfaces in $\mathbb{C}P^3$. $-4+4=0$.

② double cover of $\mathbb{P}^2$ branched along a sextic curve.

triple cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along a $(3,3)$ curve.

$$X_0 X_1 \quad Y_0 Y_1$$

$$X_0^3 Y_0^3 + X_0 X_1^2 Y_1^3 + X_1^3 Y_0^2 Y_1$$

③ Elliptic fibrations over $\mathbb{P}^1$.

Rmk: If S is a K3 surface, $S \to \mathbb{P}^1$ an elliptic fibration, then this fibration has exactly 24 branched points. (counting with multiplicities)

④ Complete intersection.

A "good" intersection of a quadric hypersurface and a cubic hypersurface in $\mathbb{P}^4$ is a K3 surface. $-5+2+3=0$

A "good" intersection of three quadric hypersurfaces in $\mathbb{P}^5$ is a K3 surface. $-6+2+2+2=0$.

⑤ Kummer construction.

Start with a complex torus $A = \mathbb{C}^2/\Lambda$ of dim. 2,

$\imath: A \to A,\ [x] \longmapsto -[x]. \quad \forall x \in \mathbb{C}^2$

$A/_2$ has $2^4 = 16$ singularities,

$\uparrow$ each singularity is resolved by a rational line $\mathbb{P}^1$

S $\quad$ S is a K3 surface.

A alg $\Leftrightarrow$ S alg.

(6) hypersurface-, cyclic-cover-, complete-intersection - constructions

can be generalized to weighted projective space.

Calabi threefolds very important because of its role in physics and mirror symmetry

Open problem: classify all Calabi-Yau threefolds,

Example: quintic threefolds, $\quad -5+5=0.$

e.g. $x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0$, in $\mathbb{P}^4$.

Example: complete intersection of two cubic fourfolds in $\mathbb{P}^5$ is a Calabi-Yau threefold. $\quad -6+3+3=0.$

Example: $(3,3)$ - hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$.

Example: triple cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ branched along a $(3,3,3)$- hypersurface.

constructed from weighted projective space.

cubic curve, quartic surface, quintic threefold.

$n \geq 4$, hypersurface in $\mathbb{P}^{n+1}$ of degree $n+2$ is a

Calabi-Yau n-fold.

$\{$hyper-Kähler manifolds$\}$ $<$ $\{$Calabi-Yau mfds$\}$.

<u>Defn</u>: A hyper-Kähler manifold is a compact simply-connected

(of dim $2n$)

Kähler manifold X, such that $H^0(X, \Omega_X^*) = \mathbb{C}\,\omega$

where $\omega \in H^0(X, \Omega_X^2)$ is a holomorphic <u>symplectic</u> form.

$\wedge^n \omega$ nowhere vanishes

Calabi-Yau threefolds: easy to construct,

"few" general theory, we do not know whether finitely many

hyper-Kähler manifolds: "hard" to construct,

"rich" geometric structures and theory, we do not know

whether finitely many.

<u>Examples</u>: 2-dim: K3 surfaces.

$\forall n \in \mathbb{N}^+$, $2n$-dim: two series (Beauville, $\cdots$):

<u>S K3</u>, $\mathrm{Hilb}^n(S)$: Hilbert space of n points on S,

a point here $\Leftrightarrow$ n non-ordered points on S,

$\mathrm{Hilb}^n(S)$ is a hyper-Kähler manifold of dim $2n$.

Generalized Kummer Construction:

Start with A: torus of dimension 2,

$$\mathrm{Hilb}^{n+1}(A) \xrightarrow[\text{take sum}]{} A.$$

$K^n(A)$ = Kernel of the above map.

this is also a hyper-Kähler mfd of dim $2n$.

Two exceptional cases: found by O'Grady.

$OG_6 \quad OG_{10}$.

Thm: Deforming complex structure on a hyper-Kähler $\rightsquigarrow$ still hyper-Kähler.

These are all known examples up to now!

Hodge diamond.

X: CY threefold.

$$H^3(X) = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}.$$

$$1 \qquad h^{2,1} = h^{1,2} \qquad 1$$

$$H^2(X) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}.$$

$$h^{1,1}$$

$H^1(X) = 0$ if X is simply connected.

Period map

Every compact Kähler manifold has a Hodge structure.

$$H^n = \bigoplus_{p+q=n} H^{p,q} \qquad \overline{H^{p,q}} = H^{q,p}.$$

Let $\begin{array}{c}\mathcal{X}\\ \downarrow\\ B\end{array}$ be a "good" family of Kähler mflds,
(each fiber is a Kähler manifold)

$B \longrightarrow$ "Moduli of Hodge structures".

Fix $b_0 \in B$, $H^*(\mathcal{X}_{b_0})$.

for $b \in B$, take a path from b to b_0,

$\rightsquigarrow$ $H^*(\mathcal{X}_b) \xrightarrow{\simeq} H^*(\mathcal{X}_{b_0})$.

H.S. $\longmapsto$ a H.S. here.

$\mathbb{D}$: space of H.S. of "this type" on $H^*(\mathcal{X}_{b_0})$.

Γ: image of $\pi_1(B, b_0) \to \mathrm{Aut}(H^*(\mathcal{X}_{b_0}))$

then we have a well-defined map $\mathcal{P}: B \to \mathbb{D}/\Gamma$.

called the period map of $\begin{array}{c}\mathcal{X}\\ \downarrow\\ B\end{array}$.

$\mathbb{D}/\Gamma$ is called the period domain.

Say we have global Torelli theorem for $\begin{array}{c}\mathcal{X}\\ \downarrow\\ B\end{array}$, if $\mathcal{P}$ is injective.

In our course, we will prove "global Torelli for K3 surface"

We also aim to talk about Verbitsky, et al 's "global Torelli" for certain hyper-Kähler manifolds

Connection, holonomy group.

De Rham, Berger, decomposition theorem of Riemannian mflds.

Let M be a manifold, $E \to M$ a vector bundle,

a connection ∇ on E is a linear map:

$$\nabla : C^\infty(E) \to C^\infty(E \otimes T^*M) \quad \text{satisfying:}$$

$$\nabla(\alpha e) = \alpha \nabla(e) + e \otimes d\alpha.$$ Libnitz rule.

where $e \in C^\infty(E)$, $\alpha \in C^\infty(M)$.

$C^\infty(E)$: set of smooth sections of E over open subset of M

Another interpretation of connection:

take $X \in C^\infty(TM)$, over U. ∇_X: {smooth sections of E over U} $\to$ {smooth sections of E over U},

$s \in C^\infty(E)$ over U.

$$\nabla_X(s) = \nabla(s) \cdot X \in C^\infty(E).$$

connection ∇, $\to$ "linear map" $T(M) \to \mathrm{Diff}(E)$

Libnitz rule for the second interpretation:

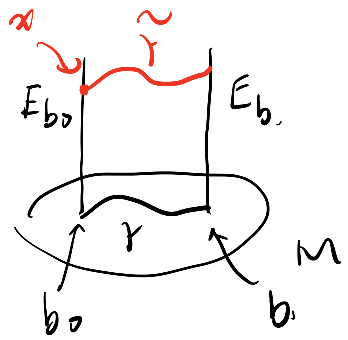
$$X \in C^\infty(TM), \quad \nabla_X(\alpha e) = \alpha \nabla_X e + (X\alpha) \cdot e.$$

$$\alpha \in C^\infty(M), \quad e \in C^\infty(E).$$

Start with $\begin{array}{c} E \\ \downarrow \\ M \end{array}$ and a connection ∇.

fix a fiber E_{b_0} for $b_0 \in M$.

take any fiber E_b for $b \in M$.



take a smooth path $\gamma : [0,1] \to M$,

$\gamma(0) = b_0$, $\gamma(1) = b$,

The connection ∇ naturally induces a connection on $E|_\gamma$.

still denoted by ∇.

Thm: $\forall\ x \in E_{b_0}$, there exists a unique smooth section

$\tilde{\gamma}$ of $E|_\gamma$, such that: $\tilde{\gamma}(0) = x$.

$$\nabla_{\underbrace{\gamma'(t)}_{\text{tangent vector field on } \gamma.}} \tilde{\gamma}(t) \equiv 0.$$

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then $\tilde{\gamma}(1)$ is called a parallel transportation of

$x = \tilde{\gamma}(0)$.

We obtain a linear map $E_{b_0} \to E_b$ from ∇.

If we take the group of loops based on b_0,

$\rightsquigarrow$ a bunch of linear isomorphisms $E_{b_0} \to E_{b_0}$.

they form a subgroup of $GL(E_{b_0})$.

called the holonomy group of (E, ∇).