Lecture 1. Introduction

Aim: Describe some connections of hypergeometric functions with KZ equations and representation theory of quantum group, via example of 3 points on $\mathbb{C}$.

§1. Examples of Hypergeometric functions (3 points on $\mathbb{C}$)

- z_1, z_2, z_3, 3-pts on $\mathbb{C}$
- Fix $m_1, m_2, m_3 \in \mathbb{C}$, $\kappa \in \mathbb{C}^*$

Consider a multivalued hol fct on $t \in \mathbb{C} - \{z_1, z_2, z_3\}$

$$\Phi = \prod_{1 \le i < j \le 3} (z_i - z_j)^{m_i m_j / 2\kappa} \prod_{j=1}^{3} (t - z_j)^{-m_j/\kappa}$$

and set

$$\eta_j = \Phi \cdot \frac{dt}{t - z_j} \quad j = 1, 2, 3$$

1) closed hol 1-form on $\mathbb{C} - \{z_1, z_2, z_3\}$

2) Cohomological relation

$$m_1 \eta_1 + m_2 \eta_2 + m_3 \eta_3 = -\kappa \, d\Phi$$

Formally, do integration over some 1-cycle $\gamma \longrightarrow H_1(\mathbb{C} - \{z_1, z_2, z_3\}; ?)$

$\rightsquigarrow I^{(\gamma)} = (I_1, I_2, I_3), = (\int_\gamma \eta_1, \int_\gamma \eta_2, \int_\gamma \eta_3)$ (formal

2) $\Longrightarrow$ $m_1 I_1 + m_2 I_2 + m_3 I_3 = 0$. discussion)

The experience tells us (or We hope) the integrals remain unchanged if we deform γ continuously. (Stokes thm). The fct $I^{(\gamma)}(z)$ is hol in a small neighborhood of their initial positions.

$$\rightsquigarrow \partial I^{(\gamma)} / \partial z_i$$

Thm. $I^{(\gamma)}$ satisfies the following equations

$$\frac{\partial I}{\partial z_i} = \sum_{j \ne i} \frac{1}{\kappa} \cdot \frac{\Omega_{ij}}{z_i - z_j} I \qquad i = 1, 2, 3$$

where

$$\Omega_{ij} = \left(\begin{array}{c|ccc} & i & \cdots & j \\ i & \frac{(m_i-2)m_j}{2} & \cdots & m_j \\ \vdots & \vdots & & \vdots \\ j & m_i & \cdots & \frac{m_i(m_j-2)}{2} \\ & \vdots & & \vdots \end{array} \right) \qquad \begin{array}{l} (\Omega_{ij})_{kk} = \frac{m_i m_j}{2} \quad k \ne i, j \\ \text{other entries are } 0. \end{array}$$

Pf. $$\frac{\partial I_2}{\partial z_1} = \int_\gamma \frac{\partial \eta_2}{\partial z_1} = \int_\gamma \Big(\frac{m_1 m_2}{2\kappa} \frac{1}{z_1 - z_2} + \frac{m_1 m_3}{2\kappa} \frac{1}{z_1 - z_3} + \frac{m_1}{\kappa} \frac{1}{t - z_1} \Big) \Phi \frac{dt}{t - z_2}$$

$$\frac{1}{t - z_1} \frac{1}{t - z_2} = \frac{1}{z_1 - z_2} \Big(\frac{1}{t - z_1} - \frac{1}{t - z_2} \Big)$$

$$= \frac{m_1(m_2 - 2)}{2\kappa} \frac{1}{z_1 - z_2} \int_\gamma \eta_2 + \frac{m_1 m_3}{2\kappa} \frac{1}{z_1 - z_3} \int_\gamma \eta_2 + \frac{m_1}{\kappa} \frac{1}{z_1 - z_2} \int_\gamma \eta_1 \qquad \#$$

<u>Rmk</u>. The combinatorial identity between t, z_1, z_2 here has deep connections w/ Jacobi identity in Lie alg thy. We can also understand it via Arnold relation, that is,

$$u_{ij} \wedge u_{jk} + u_{jk} \wedge u_{ki} + u_{ki} \wedge u_{ij} = 0, \quad \text{where} \quad u_{ij} = \frac{dz_i - dz_j}{z_i - z_j}.$$

Such $\{u_{ij}\}_{1 \leq i < j \leq n}$ generate $H^*(\mathrm{Conf}_n(\mathbb{C}))$ w/ Arnold relation.

§ KZ equations

- $\mathfrak{g} = sl_2$ generated by e, f, h w/
$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f$$
- $\Omega = \frac{1}{2} h \otimes h + e \otimes f + f \otimes e$, an invariant symm 2-tensor
i.e. $\Omega_{21} = \Omega$, $[\Omega, x \otimes 1 + 1 \otimes x] = 0 \quad \forall x \in \mathfrak{g}$
- $M_1, \ldots, M_n$, $\mathfrak{g}$-mod
via $\Delta: \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ $\quad \Delta(x) = x \otimes 1 + 1 \otimes x$
$M = M_1 \otimes \cdots \otimes M_n$ admits a $\mathfrak{g}$-mod structure.
- Ω_{ij}, linear operator on M, acting as Ω on $M_i \otimes M_j$, id on M_k, $k \neq i, j$.
$\circledast \quad [\Omega_{ij}, \Omega_{ik} + \Omega_{jk}] = 0.$

The KZ equation on M-valued fct $\varphi(z_1, \ldots, z_n)$ is given by

$$\frac{\partial \varphi}{\partial z_i} = \frac{1}{\kappa} \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} \varphi \qquad i = 1, \ldots, n$$

κ is the parameter of this equation. sometimes, write $\hbar = 1/\kappa$.

Consider $X_n = \mathrm{Conf}_n(\mathbb{C}) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_1, \ldots, z_n \text{ are distinct}\}$

$X_n \times M \longrightarrow X_n$ trivial bundle

$\nabla = d - \Gamma$ where $\Gamma = \frac{1}{\kappa} \sum_{j \neq i} \Omega_{ij} \, d\log(z_i - z_j)$

The Arnold relation + relation for $\circledast$

$\Longrightarrow \nabla^2 = 0.$

Some background of KZ eqs.

In CFT, one considers Riemann surfaces (RS) w/ marked pts decorated by rep of a Lie alg. To such a surface one assigns a finite dim'l vector space of conformal blocks.

$\rightsquigarrow$ a bundle over the moduli space of decorated surfaces

(Knizhnik- Zamolodchikov - Bernard)

$\rightsquigarrow$ KZ or KZB connection on this bundle

(When genus $= 0$, it's a flat cnn ; genus $= 1$, projectively flat.)

$\rightsquigarrow$ The horizontal section equation : KZ eq.

Q : How to study such a system of eq ?

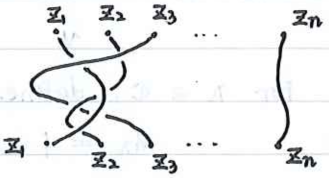
1) From the point of view of representation M :

Note that $[\Omega, \Delta(x)] = 0$, so if φ is a solution, then $x \cdot \varphi$ is also a solution.

$\rightsquigarrow$ $X_n \times M \to X_n$ has a lot of subbundles w. r. t. ∇.

e.g. eigenspaces of generators of $\mathfrak{g}$ $\uparrow$

2) From the point of view of $X_n = \mathrm{Conf}_n(\mathbb{C})$:



$\pi_1(\mathrm{Conf}_n(\mathbb{C})) = P_n$ pure braid group

Since ∇ is flat, then the analytic continuation along a loop $\alpha \in L\,\mathrm{Conf}_n(\mathbb{C})$ only dep on $[\alpha] \in \pi_1(\mathrm{Conf}_n(\mathbb{C}))$.

$\rightsquigarrow$ $\rho_{KZ} : P_n \to \mathrm{Aut}(Sol)$

In particular,

$M = M_1^{\otimes n}$ admits a left S_n-action. One can def $X_n \times M \curvearrowleft S_n$:

$((z_1, \dots, z_n), \phi) \cdot \sigma = ((z_{\sigma(1)}, \dots, z_{\sigma(n)}), \sigma^{-1}\phi)$

Then the composition $X_n \times M \to X_n \to Y_n := X_n / S_n$ factors through the quotient space $E = X_n \times_{S_n} M$.

Now, the KZ eq is invariant under S_n, then ∇ descendants to a connection on $E \to Y_n$.

$\pi_1(Y_n) = \pi_1(\mathrm{Conf}_n(\mathbb{C}) / S_n) = B_n$ braid group

$\rightsquigarrow$ $\rho_{KZ} : B_n \to \mathrm{Aut}(Sol)$

About computation of such a monodromy representation : solve equation around each singular point z_i.

($n \geq 3$ $\rightsquigarrow$ Φ_{KZ} : KZ associator) non-comm between Ω_{ij}, Ω_{jk}

$(z_i \to z_j)$ z_k , z_i $(z_j \to z_k)$

The asymptotic solutions in different zones will be related via pentagon relation, hexagon relation. ⤳ quasi-triangular quasi-Hopf str.

(One can also consider a combinatorial construction of the monodromy rep via a suitable compactification of conf space.)

3) Solution problems (hypergeometric solutions)

Now, let us briefly discuss the rep thy of sl_2.

$m \in \mathbb{C}$, $M(m)$: Verma mod over sl_2 w/ highest weight m. It's a ∞ dim'l mod generated by a single vector v w/

$$ev = 0, \quad hv = mv.$$

Take $M = \underset{v_1}{M(m_1)} \otimes \underset{v_2}{M(m_2)} \otimes \underset{v_3}{M(m_3)}$ $\qquad m = m_1 + m_2 + m_3$

For $\lambda \in \mathbb{C}$, define the eigenspace of h

$$M_\lambda = \{v \in M \mid h \cdot v = \lambda v\}$$

$$\mathrm{Sing}\, M_\lambda = \{v \in M_\lambda \mid e \cdot v = 0\}$$

(Rmk: The action of sl_2 on $\mathrm{Sing}\, M_\lambda$ generates the whole M.)

① M_m (here $=$ $\mathrm{Sing}\, M_m$) 1 dim'l, spanned by $v_1 \otimes v_2 \otimes v_3$

Recall $\Omega = \frac{1}{2} h \otimes h + e \otimes f + f \otimes e$

$$\Omega_{ij}: v_1 \otimes v_2 \otimes v_3 \longmapsto \frac{m_i m_j}{2} v_1 \otimes v_2 \otimes v_3$$

Thus the KZ eq

$$\frac{\partial \varphi}{\partial z_i} = \frac{1}{\kappa} \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} \varphi \qquad \text{for } \varphi = I_0(z) v_1 \otimes v_2 \otimes v_3$$

is just

$$\frac{\partial I_0}{\partial z_i} = \sum_{j \neq i} \frac{m_i m_j}{2\kappa} \frac{1}{z_i - z_j} I_0 \quad \rightsquigarrow \quad I_0 = \prod_{1 \leq i < j \leq 3} (z_i - z_j)^{m_i m_j / 2\kappa}$$

$\rightarrow \forall l \in \mathbb{N}_+$, $f^l \cdot I_0 v_1 \otimes v_2 \otimes v_3$ is also sol.

$l = 1$, i.e. $I_0 (f v_1 \otimes v_2 \otimes v_3 + v_1 \otimes f v_2 \otimes v_3 + v_1 \otimes v_2 \otimes f v_3)$

② $\mathrm{Sing}\, M_{m-2} = \{ v = I_1 f v_1 \otimes v_2 \otimes v_3 + I_2 v_1 \otimes f v_2 \otimes v_3 + I_3 v_1 \otimes v_2 \otimes f v_3 \mid m_1 I_1 + m_2 I_2 + m_3 I_3 = 0 \}$.

One can also compute the action of Ω_{ij}, e.g.

$$\Omega_{12}: f v_1 \otimes v_2 \otimes v_3 \longmapsto \frac{(m_1 - 2) m_2}{2} f v_1 \otimes v_2 \otimes v_3 + m_1 v_1 \otimes f v_2 \otimes v_3$$

$$v_1 \otimes f v_2 \otimes v_3 \longmapsto \frac{m_1 (m_2 - 2)}{2} v_1 \otimes f v_2 \otimes v_3 + m_2 f v_1 \otimes v_2 \otimes v_3$$

$$v_1 \otimes v_2 \otimes f v_3 \longmapsto \frac{1}{2} m_1 m_2 \, v_1 \otimes v_2 \otimes f v_3 .$$

Now LHS of KZ eq

$$= \frac{\partial I_1}{\partial z_1} f v_1 \otimes v_2 \otimes v_3 + \frac{\partial I_2}{\partial z_1} v_1 \otimes f v_2 \otimes v_3 + \frac{\partial I_3}{\partial z_1} v_1 \otimes v_2 \otimes f v_3$$

If we focus on $\frac{\partial}{\partial z_1}$ and the fct part of $v_1 \otimes f v_2 \otimes v_3$ on the RHS of KZ (i.e. $\frac{1}{\kappa} \left(\frac{\Omega_{13}}{z_1 - z_3} + \frac{\Omega_{12}}{z_1 - z_2} \right) v$)

$$\rightsquigarrow \frac{\partial I_2}{\partial z_1} = \frac{1}{\kappa} \frac{m_1}{z_1 - z_2} I_1 + \frac{1}{2\kappa} \frac{m_1(m_2 - 2)}{z_1 - z_2} I_2 + \frac{m_1 m_3}{2\kappa} \frac{1}{z_1 - z_3} I_2 .$$

Which is the same as the hypergeometric diff eq for $\partial I_2 / \partial z_1$.

One can check $I = (I_1, I_2, I_3)$ satisfies the hypergeom. diff eq.

In other words,

$$\int_\gamma \eta_1 \, f v_1 \otimes v_2 \otimes v_3 + \int_\gamma \eta_2 \, v_1 \otimes f v_2 \otimes v_3 + \int_\gamma \eta_3 \, v_1 \otimes v_2 \otimes f v_3$$

is a sol of KZ eq, taking values in Sing M_{m-2}.

Set $\mathcal{H}' = (\mathbb{C}\eta_1 \oplus \mathbb{C}\eta_2 \oplus \mathbb{C}\eta_3) / \mathbb{C} d\Phi$, then the above argument says

$$\mathcal{H}' \cong (\mathrm{Sing}\, M_{m-2})^*$$

The above solution depends on the choice of integration cycle. Now let us see what γ is.

§3. Cycles of Integrals

Since Φ is multi-valued, to integrate it over a good loop γ, we have to fix a uni-valued branch of Φ over γ.

$\longrightarrow$ homology gp of $\mathbb{C} - \{z_1, z_2, z_3\}$ w/ twisted coefficient corresp to Φ.

i.e.

Φ def a 1d local system S over $\mathbb{C} - \{z_1, z_2, z_3\}$. The sections of S are $\mathbb{C}$-linear combinations of univalued branches of Φ.

The local system has monodromy around each z_a, $e^{-2\pi i m_a / \kappa}$

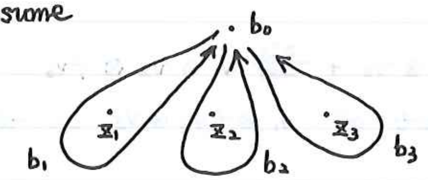
Set $q = e^{2\pi i/\kappa} \longrightarrow q^{-m_a}$.

For $\gamma \in H_1(\mathbb{C} - \{z_1, z_2, z_3\}, S)$, $\int_\gamma \eta_i$ is well defined: i.e. integration defines a pairing

$$I[z_1, z_2, z_3] : \mathcal{H}' \otimes H_1(\mathbb{C} - \{z_1, z_2, z_3\}, S) \longrightarrow \mathbb{C}$$

△ Computation of $H_1(\mathbb{C}-\{z_1, z_2, z_3\}, S)$

Assume



Choose b_0, b_1, b_2, b_3 as the picture
Fix sections s_0, s_1, s_2, s_3 of S over b_0, b_1, b_2, b_3.

The pair (b_j, s_j) is a singular chain w/ coeff in S

We need to compute

$$d, \ \oplus_{j=1}^{3}\mathbb{C}(b_j, s_j) \longrightarrow \mathbb{C}(b_0, s_0)$$

e.g. fix a value s_0 of $(t-z_1)^{-m_1/\kappa}$ at b_0,

fix a univalued branch s_1 of $(t-z_1)^{-m_1/\kappa}$ over b_1, whose value at b_1's end pt is s_0.

Then $d(b_1, s_1) = (b_0, s_1|\text{end pt}) - (b_0, s_1|\text{starting pt})$

$$= (b_0, s_0) - (b_0, q^{m_1}s_0) = (1-q^{m_1})(b_0, s_0).$$

△ Topological monodromy

Consider vector bundle over $\mathrm{Conf}_3(\mathbb{C})$ w/ fiber at (z_1, z_2, z_3) given by $H_1(\mathbb{C}-\{z_1, z_2, z_3\}, S)$. This bundle has a flat cnn (GM cnn).

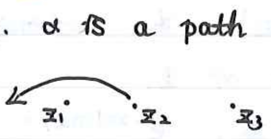
For any continuous path $[0,1] \longrightarrow \mathrm{Conf}_3(\mathbb{C})$ $\quad t \mapsto z(t)$

$\gamma^0 \in H_1(\mathbb{C}-\{z(0)\}, S) \rightsquigarrow ! \ \gamma^t \in H_1(\mathbb{C}-\{z(t)\}, S)$ via continuously deforming singular cells and analytically continuing the univalued branches over deformation of cells.

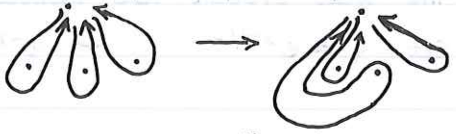
$\rightsquigarrow \rho^{top}: \pi_1(\mathrm{Conf}_3(\mathbb{C}), z(0)) \longrightarrow GL(H_1(\mathbb{C}-\{z(0)\}, S))$

Similarly $\rho^{top}: \pi_1(\mathrm{Conf}_3(\mathbb{C})/S_3) \longrightarrow GL(H_1(\mathbb{C}-\{z(0)\}, S))$

e.g. α is a path by moving z_2 in front of z_1



Then $\rho^{top}(\alpha)$ is as follows



Let $\gamma(z) \in H_1(\mathbb{C}-\{z\}, S)$ be the flat section of ∇^{GM} over a nbhd of $z(0)$ st $\gamma(z(0)) = \gamma^0$. Then

$$\phi^{(T)}(z_1, z_2, z_3) = \sum_{i=1}^{3} \int_{\gamma(z)} \eta_i \cdot f^{(i)} v_1 \otimes v_2 \otimes v_3.$$

gives a KZ eq w/ values in Sing M_{m-2}, over a nbhd of $z(0)$.

$\rightsquigarrow$ $H_1(\mathbb{C}-\{z(0)\}, S) \longrightarrow Sol_{z(0)}$ ($\cong$ Sing M_{m-2}) homomorphism

Thm. It's a homomorphism of P_3 - mod.

It's an isomorphism for generic κ, m_1, m_2, m_3 !

(via determinant formula)

Thm. $\exists$ natural choice of sections s_0, s_1, s_2, s_3 st the boundary op has the following form.

$$(b_1, s_1) \longmapsto (q^{m_1/2} - q^{-m_1/2})\, q^{(m_2+m_3)/4} (b_0, s_0)$$
$$(b_2, s_2) \longmapsto (q^{m_2/2} - q^{-m_2/2})\, q^{(-m_1+m_3)/4} (b_0, s_0)$$
$$(b_3, s_3) \longmapsto (q^{m_3/2} - q^{-m_3/2})\, q^{(-m_1-m_2)/4} (b_0, s_0).$$

Such a special choice of generators establishes a connection w/ quantum gp.

§4. Quantum group $U_q sl_2$

- $U_q sl_2$, a $\mathbb{C}$-alg genenerated by e, f, h w/
$$[e, f] = q^{h/2} - q^{-h/2}$$
$$[h, e] = 2e, \quad [h, f] = -2f$$
It's a Hopf alg, where the coproduct is given by
$$\Delta(h) = h \otimes 1 + 1 \otimes h$$
$$\Delta(f) = f \otimes q^{h/4} + q^{-h/4} \otimes f$$
$$\Delta(e) = e \otimes q^{h/4} + q^{-h/4} \otimes e$$

- Verma module $M(m, q)$ $\quad ev = 0, \ hv = mv$
- $M(q) = M(m_1, q) \otimes M(m_2, q) \otimes M(m_3, q)$ $\quad m = m_1 + m_2 + m_3$

similarly, $M(q)_\lambda$, Sing $M(q)_\lambda$

e.g. Sing $M(q)_{m-2} = \{ v \in M(q) \mid \underline{\underline{ev = 0}},\ hv = (m-2)v \}$

$= \{ v = \Sigma_{i=1}^{3} I_i f^{(i)} v_1 \otimes v_2 \otimes v_3 \mid \updownarrow$

$(q^{m_1/2} - q^{-m_1/2})\, q^{(m_2+m_3)/4} I_1 + (q^{m_2/2} - q^{-m_2/2})\, q^{(-m_1+m_3)/4} I_2$

$+ (q^{m_3/2} - q^{-m_3/2})\, q^{(-m_1-m_2)/4} I_3 = 0 \}$

The map $(b_i, s_i) \longmapsto f^{(i)} v_1 \otimes v_2 \otimes v_3$ defines the natural isom

$$\oplus_{j=1}^{3} \mathbb{C}(b_j, S_j) \cong M(q)_{m-2}$$

$$\ker d \longleftrightarrow ev = 0$$

$$\rightsquigarrow H_1(\mathbb{C}-\{z_1, z_2, z_3\}, S) \cong \mathrm{Sing}\, M(q)_{m-2}.$$

⊛ $\mathcal{H}' \cong (\mathrm{Sing}\, M_{m-2})^*$, $H_1(\mathbb{C}-\{z_1, z_2, z_3\}, S) \cong \mathrm{Sing}\, M(q)_{m-2}$

the integration induces the pairing

$I[z_1, z_2, z_3]: (\mathrm{Sing}\, M_{m-2})^* \otimes \mathrm{Sing}\, M(q)_{m-2} \longrightarrow \mathbb{C}$ hypergeom pairing

For generic κ, m_1, m_2, m_3, it's non-degenerate.

$\rightsquigarrow J[z_1, z_2, z_3]: \mathrm{Sing}\, M(q)_{m-2} \longrightarrow \mathrm{Sing}\, M_{m-2}$

$\forall v \in \mathrm{Sing}\, M(q)_{m-2}$,

$(z_1, z_2, z_3) \longmapsto J(z_1, z_2, z_3)(v)$

def a solution of KZ w/ values in $\mathrm{Sing}\, M_{m-2}$.

△ R-matrix representation

Recall that the mod M_i "lives at z_i", w/ z_i's moving, we sometimes have to interchange the positions of M_i and M_j in $\otimes$. But unlike in the classical thy, if V, W are $U_q sl_2$-mod,

$$P: V\otimes W \longrightarrow W \otimes V \qquad v\otimes w \longmapsto w \otimes v$$

is not an isom of rep.

To obtain a isom of rep, one needs the R-matrix

$$R \in U_q sl_2 \,\hat{\otimes}\, U_q sl_2$$

$$V\otimes W \xrightarrow{R} V \otimes W \xrightarrow{P} W \otimes V$$

Thm. V_1, V_2, V_3 are $U_q sl_2$-mod. Then

$$\begin{tikzcd}
V_1 \otimes V_2 \otimes V_3 \arrow[r, "(23)"'] \arrow[d, "(12)"'] & V_1 \otimes V_3 \otimes V_2 \arrow[r, "(13)"'] \arrow[d, phantom, "\circlearrowright"] & V_3 \otimes V_1 \otimes V_2 \arrow[d, "(23)"] \\
V_2 \otimes V_1 \otimes V_3 \arrow[r, "(23)"] & V_2 \otimes V_3 \otimes V_1 \arrow[r, "(12)"] & V_3 \otimes V_2 \otimes V_1
\end{tikzcd}$$

(Comm $\Longleftrightarrow$ YBE)

In general, let $V_1, \dots, V_n$ be $U_q sl_2$-mod. Let

$$R_i^{\vee}: V_1 \otimes \cdots \otimes V_i \otimes V_{i+1} \otimes \cdots \otimes V_n \longrightarrow V_1 \otimes \cdots \otimes V_{i+1} \otimes V_i \otimes \cdots \otimes V_n$$

Then $R_i^\vee R_{i+1}^\vee R_i^\vee = R_{i+1}^\vee R_i^\vee R_{i+1}^\vee : V_1 \otimes \cdots \otimes V_i \otimes V_{i+1} \otimes V_{i+2} \otimes \cdots \otimes V_n$

$\longrightarrow V_1 \otimes \cdots \otimes V_{i+2} \otimes V_{i+1} \otimes V_i \otimes \cdots \otimes V_n$

$\rightsquigarrow \rho_R : P_n \longrightarrow \mathrm{Aut}\,(V_1 \otimes \cdots \otimes V_n) \qquad b_i \mapsto R_i^\vee$.

if $V_i = W$ all i, then

$\rho_R : B_n \longrightarrow \mathrm{Aut}\,(W^{\otimes n})$

Rmk. $R_i^\vee$ preserves the weight decomposition of $\otimes V_i$ and the subspace of singular vectors.

Thm. $H_1(\mathbb{C} - \{z_1, z_2, z_3\}, S') \cong \mathrm{Sing}\, M(q)_{m-2}$ as a P_3-mod.

In summary, we have 3 kinds of rep of P_3

$\rho_{KZ} : P_3 \longrightarrow \mathrm{Aut}\,(\mathrm{Sol}) \quad (\cong \mathrm{Aut}\,(\mathrm{Sing}\, M_{m-2}))$

$\rho_{top} : P_3 \longrightarrow GL\,(H_1(\mathbb{C} - \{z_1, z_2, z_3\}, S'))$

$\rho_R : P_3 \longrightarrow \mathrm{Aut}\,(\mathrm{Sing}\, M(q)_{m-2})$

The above argument says for generic κ, m, they are equivalent.

Some generalizations :

Consider

$$\Phi_{k,n}(t, z, m) = \prod_{1 \le i < j \le n} (z_i - z_j)^{\frac{m_i m_j}{2\kappa}} \prod_{1 \le i < j \le k} (t_i - t_j)^{\frac{2}{\kappa}} \prod_{l=1}^{n} \prod_{i=1}^{k} (t_i - z_l)^{-m_l/\kappa}$$

as a fct of t.

It's a multi-valued fct on

$$C_{k,n}(z) = \{t \in \mathbb{C}^k \mid t_i \neq z_l,\ t_i \neq t_j,\ \forall\ i, j, l\}$$

Q : How to asso. a form like η_1 to $\Phi_{k,n}$?

What is $H_*(C_{k,n}(z), S')$?

What is the corresponding mod ?