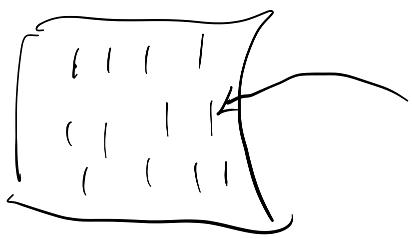
# Calabi-Yau manifolds in Algebraic Geometry.

"Ricci flatness" space.

theoretical physics, espec. string theory.



$\mathbb{R}$-dim

6-dim'l manifolds.

Calabi-Yau threefolds.

$\mathbb{C}$-dim.

⇝ Mirror symmetry.

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Algebraic Geometry ⇝ a lot of examples of Kähler manifolds, also Calabi-Yau manifolds.

"algebraic object".

essentially constructed from polyno.

Although "algebraicity" is very restrictive,

Alg. Geo. offers "many" Calabi-Yau mfds up to deformation of complex structures.

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~1954 Calabi conjectured the existence of such "Ricci flat" space.

1978. Shing-Tung Yau proved Calabi conjecture, i.e. existence of such space.

**Defn.** A Calabi-Yau n-fold or Calabi-Yau manifold of (complex) dimension $n$ is defined as a compact $n$-dimensional Kähler manifold $M$ satisfy one of the following equivalent conditions:

① • The canical bundle $K_M$ of $M$ is trivial;

② • $M$ has a holomorphic $n$-form vanishes nowhere;

$\Updownarrow$

• ③ $M$ has a Kähler metric with holonomy group contained in $SU(n)$

highly nontrivial.

① ⇒ ③ : Yau's theorem.

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Some authors allow a weaker definition of Calabi-Yau manifolds:

They define a Calabi-Yau manifold as a compact $n$-dim'l Kähler manifold $M$, satisfying one of the following equivalent conditions.

①. $M$ has vanishing first real Chern class (i.e. $c_1(K_M)=0$)

② $M$ has a Kähler metric with vanishing Ricci curvature

③ $M$ has a Kähler metric with local holonomy group contained in $SU(n)$

④ A positive power of $K_M$ is trivial.

⑤ $M$ has a finite cover that has trivial canonical bundle.

⑥ $M$ has a finite cover that is product of a torus and a simply connected manifold with trivial canonical bundle.

Example: In dimensional two,

K3 surfaces are Calabi-Yau manifolds (surface) in the stronger version  all simply connected

Enriques surface: not simply connected.

$\pi_1 = \mathbb{Z}/2$.

their universal covers are K3 surfaces.

they are regarded as Calabi-Yau manifolds in the weaker version.

Some definitions require Calabi-Yau manifolds to be simply connected, but we do not.

Examples: $\mathbb{C}^n \supset \Lambda$ discrete subgroup (under $+$)

with $\operatorname{rank}(\Lambda) = 2n$.

$\rightsquigarrow$ compact Kähler manifold $\mathbb{C}^n/\Lambda$ of dim. $n$.

$\pi_1(\mathbb{C}^n/\Lambda) = \mathbb{Z}^{2n}$ not simply connected

$dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$ descends to a holomorph n-form on $\mathbb{C}^n/\Lambda$, which is nowhere vanishing.

In our "stronger" defn, $\mathbb{C}^n/\Lambda$ is a Calabi-Yau manifold.

call torus.

Manifold $\rightsquigarrow$ non-singular.

In algebraic geometry, there may appear "singular Calabi-Yau varieties".

In most of our course, we only consider "Calabi-Yau manifolds" or "non-singular Calabi-Yau varieties".

set $\to$ topology $\nearrow$ Euclidean topology (locally homeo. to $\mathbb{R}^n$) $\searrow$ manifolds $\rightsquigarrow$ complex manifolds $\to$ Kähler manifolds $\to$ Calabi-Yau manifolds.

manifolds $\rightsquigarrow$ Riemannian manifolds $\nearrow$ Kähler manifolds

set $\to$ topology $\searrow$ Zariski topology (something from commutative algebra).

(Chow). Thm: If a ~~Kähler~~ complex manifold can be holomorphically embedded into $\mathbb{C}P^N$ (complex projective space), then it is an complex proj. algebraic variety (namely, it is cut out by homogeneous polynomials).

Conversely, let's start with $\mathbb{C}P^N$, and consider a smooth proj. subvar. $X$ of $\mathbb{C}P^N$,

then $X$ is automatically a compact complex manifold.

Actually, $X$ must be Kähler due to existence of the so-called Fubini-Study metric on $\mathbb{C}P^N$.

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<u>Examples</u>: <u>dim'e 1</u>, <u>$\mathbb{C}/\Lambda$</u>: torus.

<u>smooth cubic curves in $\mathbb{C}P^2$</u>:

$$\mathbb{C}P^2 = \mathbb{C}^3 - \{0\} / \mathbb{C}^\times = \{ [x_1 : x_2 : x_3] \,/\, x_1, x_2, x_3 \in \mathbb{C}, \text{ at least one nonzero} \}$$

$[x_1 : x_2 : x_3] = [\lambda x_1 : \lambda x_2 : \lambda x_3]$ for $\lambda \in \mathbb{C}^\times$.

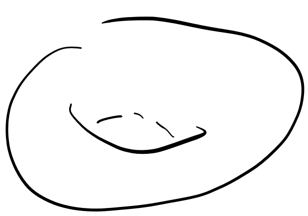
$F(x_1, x_2, x_3)$ cubic polynomial,

e.g. $x_1^3 + x_2^3 + x_3^3$.

$$Z(F) = \{ [x_1 : x_2 : x_3] \in \mathbb{C}P^2 \mid F(x_1, x_2, x_3) = 0 \}.$$

is a well-defined subset of $\mathbb{C}P^2$.

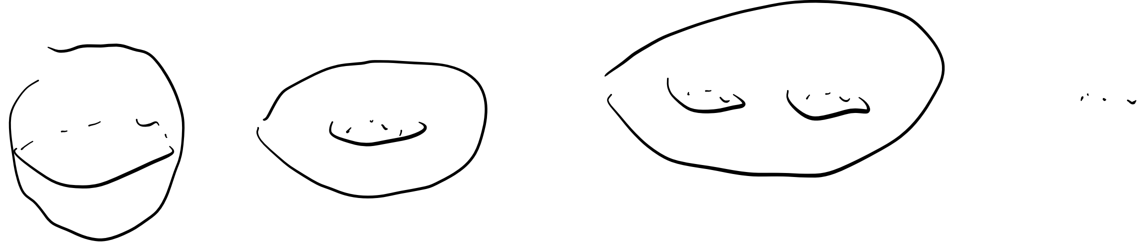
is called a cubic curve in $\mathbb{C}P^2$.



$\longleftrightarrow$ smooth cubic curves in $\mathbb{C}P^2$.

$\mathbb{C}/\Lambda$

Compact topological manifold of real dim. two.



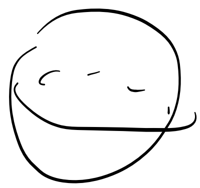
genus = 0    1    2.

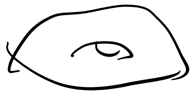
compact complex manifold of $\mathbb{C}$-dim one (Riemann surface)

they are all algebraic. (then called complex algebraic curve)

compact Riemann surface has genus.

If genus = 0, can only be $\mathbb{CP}^1$. (Riemann sphere)



If genus = 1.  torus.  cubic curve in $\mathbb{CP}^2$.

If genus ≥ 2. $C$ $g(C) \geq 2$. $K_C$ is non-trivial.

not Calabi-Yau.

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genus = 0 $\mathbb{CP}^1$

genus = 1 torus.

genus ≥ 2. hyperbolic curve. (can be embedded with a Riemannian metric with constant curvature $-1$)

$$g(C) \geq 2. \quad \mathbb{H} \leftrightarrow \{|z|<1\} \subset \mathbb{C}, \quad \mathbb{H} = \{\operatorname{Im} z > 0\} \subset \mathbb{C}$$

$$\mathbb{H} \xrightarrow{\text{universal covering}} C.$$

the disk has Poincaré metric descends to a metric on $C$.

$K_X$ ampleness.

Fano if $K_X^{-1}$ ample. ($\mathbb{CP}^1$)

In higher dimensional, ...

of general type if $K_X$ ample $(g \geq 2)$

Thm: Calabi-Yau mfds in dim 2 must be K3 surfaces or complex tori of dimension two.

$$\mathbb{C}^2/\Lambda \qquad \pi_1 = \mathbb{Z}^4.$$

(not necessarily algebraic).

If $\mathbb{C}^2/\Lambda$ is algebraic (i.e. $\exists$ $\mathbb{C}^2/\Lambda \hookrightarrow \mathbb{CP}^N$) then it is called an abelian surface

More generally, $n \in \mathbb{N}^+$, $\mathbb{C}^n/\Lambda$ : complex torus of dim. $n$

If $\mathbb{C}^n/\Lambda$ is algebraic, then $\mathbb{C}^n/\Lambda$ is called an abelian variety of dim. $n$.

Defn. ① A complex K3 surface is a simply-connected compact complex surface $S$ with a nowhere vanishing holomorphic two form.

Defn ② A complex K3 surface is a compact complex surface $S$ with a nowhere vanishing holo. 2-form, and without a nonzero holomorphic one form. $(H^0(S, \Omega^1_S) = 0)$

① $\Rightarrow$ ② is clear.

① ② are actually equivalent.

A complex K3 surface may not be algebraic.

Kodaira. Enriques. ?? classification of compact complex surface

$\Rightarrow$ 2-dim'l Calabi-Yau mfds must be tori or K3.

K3 from algebraic geometry:

① smooth quartic surfaces in $\mathbb{C}P^3$.

$$Z(x_0^4 + x_1^4 + x_2^4 + x_3^4)$$

② Cyclic cover construction.

$\mathbb{C}P^2 \supset C$ smooth sextic curve (a curve of deg. 6)

take double cover $S \longrightarrow \mathbb{C}P^2$ branched along $C$

then $S$ is a K3 surface

$\mathbb{C}P^1 \times \mathbb{C}P^1 \supset C$ curve of type (3,3).

triple cover of $\mathbb{C}P^1 \times \mathbb{C}P^1$ branched along $C$ is a K3

③ K3 can be constructed as elliptic fibrations over $\mathbb{C}P^1$

④ Complete intersection. ⑤ Constructed from tori.